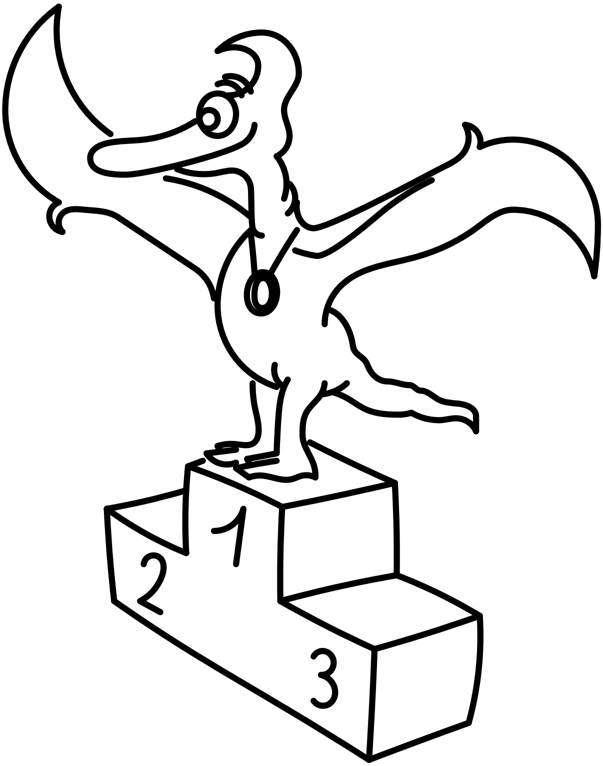
Solutions of problems

fyziklani2020



Problem AA ... cold lemonade

The FYKOS bird wanted to cool down a bottle of the Bohemsca lemonade, so he dipped it into a well. What volume would he have to drink beforehand in order to make the bottle float? The total external volume of the lemonade bottle is $0.420l$ *and the maximum volume of a liquid that can fit inside it is* $0.336\,\mathrm{l}$. *The original volume of lemonade in a filled bottle is written on the bottle. The density of glass is* 2.5 *times greater than the density of water.*

Jáchym became thirsty while digging a well.

Let's denote the volumes described in the problem statement by V_e, V_i, $V_\mathrm{l} = 330\,\mathrm{ml}$ respectively, and also denote $k = \varrho_\mathrm{g}/\varrho_\mathrm{w} = 2.5$. The bottle floats only if its mean density is equal to that of water. The density of lemonade is almost the same as the density of water, so we only need to consider the rest - the glass and air in the bottle. The volume of glass and air together is

$$V = V_\mathrm{g} + V_\mathrm{i} - (V_\mathrm{l} - \Delta V) \,,$$

where V_g is the volume of glass and ΔV is the volume of lemonade that the FYKOS bird must have drunk. The mass of the air is insignificant, so the total mass of glass and air is $m = m_\mathrm{g} = V_\mathrm{g}\varrho_\mathrm{g}$. The bottle floats if

$$\varrho_\mathrm{w} = \frac{m}{V} = \frac{V_\mathrm{g}\varrho_\mathrm{g}}{V_\mathrm{g} + V_\mathrm{i} - (V_\mathrm{l} - \Delta V)} \,.$$

Finally, after substituting for the volume of the glass $V_\mathrm{g} = V_\mathrm{e} - V_\mathrm{i}$, we get the volume of lemonade the FYKOS bird must drink as

$$\Delta V = k\left(V_\mathrm{e} - V_\mathrm{i}\right) + V_\mathrm{l} - V_\mathrm{e} = 120\,\mathrm{ml} \,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem AB ... ropey

A new rope with a length $l_0 = 60.0\,\mathrm{m}$ *has a diameter* $d_0 = 9.40\,\mathrm{mm}$. *After some use, its diameter changes to* $d = 10.10\,\mathrm{mm}$. *Find out how much the rope shortens, assuming that its volume stays constant.*

Dodo has a new rope.

For the volume of the rope, we can use the formula for a cylinder

$$V = \frac{\pi l_0 d_0^2}{4} = \frac{\pi l d^2}{4} \,,$$

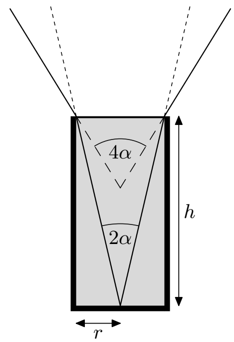
where l is the length of the used rope. Now we express this new length of the rope and calculate the difference

$$\Delta l = l - l_0 = \left(\frac{d_0}{d}\right)^2 l_0 - l_0 = \frac{d_0^2 - d^2}{d^2} l_0 \,.$$

After numerical evaluation, we get that the rope shrank by $8.0\,\mathrm{m}$. That is unexpected.

Jozef Lipták
`liptak.j@fykos.cz`

Problem AC ... at the bottom



Dano has a dry well with constant circular cross-section and depth h. When standing at the bottom, he is able to see an angle $2\alpha \leq 90^\circ$ of the sky above him. We want to increase this angle by pouring some specific liquid into the well (Dano still stays at the bottom). Find the condition for the index of refraction of the liquid if we want the angle of the visible sky to be twice as large.

Jáchym didn't know how it began...

The angle increases the most when we fill the well completely. A light ray from Dano to the top of the well inclined at an angle α from the vertical is bent by the interface in such a way that it forms the angle 2α with the vertical, in order to see double the original angle of the sky. If we denote the refractive index of the liquid by n, Snell's law implies

$$\sin 2\alpha = n \sin \alpha \,.$$

Using the double angle formula $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ for the left hand side,

$$n = 2 \cos \alpha \,.$$

That's the boundary condition, so we need n with greater or equal value,

$$n \geq 2 \cos \alpha \,.$$

The answer is independent of the dimensions of the well.

Jáchym Bártík
`tuaki@fykos.cz`

Problem AD ... snow on the roof

There is a $h = 20\,\mathrm{cm}$ (in vertical direction) thick layer of snow on our roof. The slope of the roof is $\alpha = 55^\circ$, the density of the snow is $\varrho = 0.80\,\mathrm{kg{\cdot}dm^{-3}}$ and the dimensions of the roof are $30\,\mathrm{m} \times 6\,\mathrm{m}$ (when viewed in the direction perpendicular to it). Find the pressure exerted on the roof by the snow. *Dodo was taking a shower.*

The total mass of the snow on the roof is

$$m = Sh\varrho \cos \alpha \,.$$

Pressure is defined as the perpendicular force F_n divided by the surface of the roof S

$$p = \frac{F_\mathrm{n}}{S} = \frac{mg \cos \alpha}{S} = h\varrho g \cos^2 \alpha \doteq 520\,\mathrm{Pa} \,.$$

The snow presses on the roof with $p = 520\,\mathrm{Pa}$ of pressure.

Jozef Lipták
`liptak.j@fykos.cz`

Problem AE ... Danka's glasses

The maximum distance at which Danka can see sharply with a naked eye is 20 cm*. What glasses does Danka need to see properly (i.e. so her far point is located at the correct distance in front of the eye)? Find the type and optical power of the lenses.* *Danka saw nothing.*

Let's use the thin lens equation

$$\frac{1}{a} - \frac{1}{a'} = \frac{1}{f}\,.$$

Danka requires the glasses to project the far point (the point at an infinite distance a) to the distance $a' = 20\,\mathrm{cm}$ in front of her eye, where she can see sharply. In our notation, the optical power $\Phi = -\frac{1}{f}$, so we get $\Phi = -5\,\mathrm{D}$. Danka needs concave lenses with optical power $-5\,\mathrm{D}$.

Daniela Pittnerová
daniela@fykos.cz

Problem AF ... interdimensional potential

Consider a planet with the same equatorial gravitational acceleration a_g and centrifugal acceleration a_o as the Earth ($a_\mathrm{g} = 9.83\,\mathrm{m{\cdot}s^{-2}}$ and $a_\mathrm{o} = 0.034\,\mathrm{m{\cdot}s^{-2}}$ respectively), but with a radius of only $R = 5.00\,\mathrm{km}$. How big is the difference between the gravitational potential energy of a small satellite on the surface of the planet and its potential energy infinitely far from the planet? We are interested in the absolute value of this quantity per one kilogram of the satellite's mass. *Karel was watching where Rick and Morty's family escaped to.*

The gravitational potential energy is defined to be zero at infinite distance from the source if the source is a point mass (or a sphere, which has the same field). In both cases, therefore, the energy at infinity is zero. At the finite distance R, we have the potential energy for our planet

$$E_{p1} = -G\frac{mM}{R}\,.$$

The problem statement asks for the energy per mass of our satelite, which is the gravitational potential

$$U_1 = \frac{E_{p1}}{m} = -\frac{GM}{R} = -a_g R \doteq -49\,150\,\mathrm{m^2{\cdot}s^{-2}}\,,$$

where we have used

$$a_g = \frac{GM}{R^2}$$

obtained from Newton's law of gravity.

In comparison, for the Earth with radius $R_\mathrm{Z} = 6\,380\,\mathrm{km}$, we get

$$U_2 = -a_g R_\mathrm{Z} \doteq 6.3 \cdot 10^7\,\mathrm{m^2{\cdot}s^{-2}}\,.$$

We can see that for interstellar travel, a planet with a smaller radius is better. On the contrary, for the inhabitants, a bigger planet with smaller tidal forces is better. Anyway, considering the density of this planet, which was computed at the Online Physics Brawl, it's unlikely that such a planet exists. Neutron stars have such density, but they usually have fast periods of rotation and bigger surface gravity.

Karel Kolář
karel@fykos.cz

Problem AG ... caffeine

Daniel is watching his intake of caffeine. A standard cup of coffee contains approximately 80.0 mg *of caffeine. Daniel is preparing his coffee in his moka pot, in which case one cup with volume* 1.00 dl *contains three times more caffeine than a standard cup. FYKOS has ordered new cups in the shape of equilateral cylinders – that means their height is the same as the diameter of the base, which is* 8 cm*. How much caffeine (in mg) does Daniel receive by drinking a full new cup of coffee prepared in the moka pot?* *Daniel is drinking too much coffee.*

When 1 dl of coffee from Daniel's moka pot contains three times more caffeine than a standard cup, it contains 240 mg of caffeine. We can calculate the volume of new pot using the formula for the volume of a cylinder

$$V = \frac{1}{4}\pi d^3 \doteq 402\,\mathrm{cm}^3 = 4.02\,\mathrm{dl}\,.$$

In the end, the amount of caffeine is proportional to the volume of coffee, so we get that there are 965 mg of caffeine in a full new cup.

Daniela Pittnerová
daniela@fykos.cz

Problem AH ... non-ideal voltmeter

Consider a resistor with resistance R. We connect a voltmeter in parallel with the resistor and an ammeter in series with them. Then, we connect this circuit to a DC voltage source. The ammeter shows a current I, the voltmeter a voltage U (where $U \neq RI$). Calculate the inner resistance of the voltmeter. *Legolas was measuring resistances as physics lab practice.*

The voltmeter is connected in parallel with the resistor. Therefore, U is the voltage on the resistor. The current which flows through the resistor is

$$I_\mathrm{r} = \frac{U}{R}\,.$$

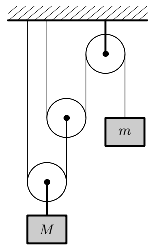
However, the ammeter is connected in series with the parallel combination of the resistor and voltmeter, so the current flowing through it is the sum of currents flowing through each of them, $I = I_\mathrm{r} + I_\mathrm{v}$.

Now, we only need to consider that the voltage shown on the voltmeter is the voltage on its probes. From the ratio of these two quantities, we can calculate the resistance

$$R_\mathrm{v} = \frac{U_\mathrm{v}}{I_\mathrm{v}} = \frac{U}{I - I_\mathrm{r}} = \frac{U}{I - \frac{U}{R}} = \frac{UR}{IR - U} = \left(\frac{I}{U} - \frac{1}{R}\right)^{-1}\,.$$

Šimon Pajger
legolas@fykos.cz

Problem BA . . . pulleys again



What is the downward acceleration of the body with mass M? Neglect the masses of all the pulleys and ropes.
We noticed that participants of Online Physics Brawl found pulleys difficult. . .

Let T be the tension in the rope on which the mass m is hanging. Then (since pulleys are massless and thus the total force acting on them must be zero), the second pulley from the left experiences an upward force $2T$. This force has to be compensated by the other rope, so the tension in it must also be $2T$.

The equations of motion for the masses are

$$Ma_M = Mg - 4T\,,$$
$$ma_m = mg - T\,,$$

where both the accelerations point downwards.

Now we have two equations with three unknowns. We need to find a relation between the accelerations. From the bare fact that the force exerted on the mass M by the rope is 4 times larger, we can assume that it will accelerate 4 times slower.

We can easily prove it geometrically. Imagine that we displace the mass M downwards by x. The second pulley must then move by $2x$ downwards, which moves m by $4x$ upwards. Thus the acceleration of the mass m is 4 times the acceleration of M (in the opposite direction), i.e. $a_m = -4a_M$.

Plugging this into our equations, we get

$$Ma_M = Mg - 4T\,,$$
$$-4ma_M = mg - T\,.$$

After solving the equations, we obtain the result

$$Ma_M + 16ma_M = Mg - 4mg\,,$$
$$a_M = g\frac{M - 4m}{M + 16m}\,.$$

For $M > 4m$, the body with mass M will accelerate downwards.

Šimon Pajger
`legolas@fykos.cz`

Problem BB . . . full tank, please

The water level in Matěj's well began to drop, so Matěj started to fill it with a water stream with constant mass flow q. Meanwhile, he noticed that just above the water level (which is $h = 37\,\mathrm{m}$ deep), the cross-section of the stream is 13 times smaller. What is the velocity of the water at the top of the well? Neglect surface tension. *Jáchym likes uncommon wells.*

Let us denote the lower cross-section by S. The upper cross-section is $S_0 = kS$, where $k = 13$. The velocities of water at the bottom and at the top are v and v_0 respectively. The mass flow rate must remain unchanged along the whole stream, therefore

$$vS = v_0S_0\,.$$

From this condition, we get $v = kv_0$. We write the equations of motion

$$v = v_0 + gt\,,$$
$$h = v_0 t + \frac{1}{2} g t^2\,,$$

express time from the first formula as

$$t = \frac{v - v_0}{g} = \frac{(k-1)\, v_0}{g}$$

and substitute it into the second formula, so we get

$$h = v_0 \frac{(k-1)\, v_0}{g} + \frac{1}{2} g \frac{(k-1)^2\, v_0^2}{g^2}\,.$$

The result is

$$v_0 = \sqrt{\frac{2gh}{k^2 - 1}} \doteq 2.1\,\mathrm{m{\cdot}s^{-1}}\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem BC ... dropped

Štěpán dropped a very heavy cannonball into a well. After the time 3.69 s*, he heard a loud "splash". How deep is the water level in the well? Neglect air resistance, but assume that the speed of sound is finite.* *10/10 made Jáchym happy.*

Let h denote the depth of the water level. The duration of the cannonball's fall is

$$t_1 = \sqrt{\frac{2h}{g}}\,.$$

The time necessary for the signal to propagate back to Štěpán is

$$t_2 = \frac{h}{c}\,,$$

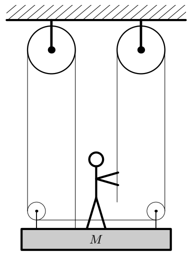
where c is the speed of sound in the air. Obviously, $t_1 + t_2 = t$, and from this, we can express the desired depth

$$0 = h^2 g - 2hc\,(tg + c) + t^2 c^2 g\,,$$
$$h = \frac{c}{g}\left((tg + c) \pm \sqrt{2tgc + c^2}\right)$$

The correct root is the one with the $-$ sign, which gives $h \doteq 60.4\,\mathrm{m}$.

Jáchym Bártík
`tuaki@fykos.cz`

Problem BD ... phy-lley



A member of FYKOS with mass $m = 50\,\mathrm{kg}$ is pulling a rope downward with a constant force $F = 300\,\mathrm{N}$. Find the magnitude of his acceleration. The mass of the depicted platform is $M = 50\,\mathrm{kg}$. Neglect the moments of inertia of all pulleys. *Matěj likes to dig holes.*

We will assume a simpler situation where the member of FYKOS and the platform form one rigid body. Later, we will explain why this assumption is correct. Between this body and the ceiling, the rope is stretched four times. A typical feature of a rope in a system of ideal pulleys is that the force of tension in it is everywhere the same. The total force that acts on the FYKOS member is $4F$. Furthermore, the force of gravity acting on it is $F_g = (M + m)\,g$. Let us use the standard formula for acceleration

$$a = \frac{4F - F_g}{M + m} = \frac{4F}{M + m} - g = 2.19\,\mathrm{m{\cdot}s^{-2}}\,.$$

The FYKOS member and the surface were assumed to be one body because apart from the surface, only the rope acts on the FYKOS member, upwards with the force $F < mg$. If he wasn't being lifted by the surface, he would start falling.

Matěj Mezera
m.mezera@fykos.cz

Problem BE ... how to move the world

Lego wanted to move the world, so he jumped down from a tree. His centre of mass shifted by h relative to the Earth. By how much did the Earth move, in the reference frame of the centre of mass of the Lego-Earth system? Lego's mass is m, the mass of Earth is M. Assume that the radius of Earth $R_\mathrm{Z} \gg h$. Do not assume anything about the masses of Lego and the Earth. *Lego wanted to move the world.*

Since $R_z \gg h$, the forces acting on both Lego and the Earth are constant during the whole fall and both have the same magnitude F. Lego will then fall with the acceleration $a_L = F/m$ towards the Earth and the Earth will fall with the acceleration $a_Z = F/M$ towards Lego. Their relative acceleration is

$$a_v = a_L + a_Z = F\left(\frac{1}{m} + \frac{1}{M}\right) = F\frac{m + M}{mM}\,.$$

We can easily calculate the duration of the fall

$$h = \frac{1}{2}a_v t^2\,,$$
$$t^2 = 2\frac{h}{a_v}\,.$$

We plug this time into the kinematic equation for uniformly accelerated motion and get the resulting displacement

$$s = \frac{1}{2}a_Z t^2 = \frac{1}{2}\frac{F}{M}2\frac{h}{F\frac{m+M}{mM}} = h\frac{m}{m + M}\,.$$

However, the displacement can be found even easier. It is sufficient to find the equation for the relative distance x of the Earth from the centre of mass of the whole system as a function of the distance d between Lego and the Earth. We find the distance from equality of torques

$$xM = (d - x)\, m\,,$$

$$x = \frac{dm}{M + m}\,.$$

During the fall, the distance d changed by h, thus the Earth moved relative to the common centre of mass by $\Delta x = \frac{hm}{M+m}$.

Šimon Pajger
legolas@fykos.cz

Jozef Lipták
liptak.j@fykos.cz

Problem BF ... a bubble in a sea

At the seabed in the depth $h_1 = 130\,\mathrm{m}$ under the sea level, a scuba diver releases an air bubble with temperature $t_1 = 36\,^{\circ}\mathrm{C}$ and radius $r_1 = 0.50\,\mathrm{cm}$. The bubble moves upwards without dividing into smaller bubbles or changing its shape. What is the radius of the bubble in the depth $h_2 = 5\,\mathrm{m}$ under the sea level? Assume that there's no heat exchange between the bubble and the sea during the ascent of the bubble. The density of the seawater is $\varrho = 1\,020\,\mathrm{kg{\cdot}m^{-3}}$, the atmospheric pressure is $p_\mathrm{a} = 1\,013\,\mathrm{hPa}$. *Danka wants to go diving.*

The bubble has a spherical shape, therefore its volume depends on its radius as $V = \frac{4}{3}\pi r^3$. Assuming no heat exchange, we are dealing with an adiabatic process; therefore, $pV^\kappa = \text{const}$, where $\kappa = 1.4$ is the adiabatic constant (ratio of heat capacities) for air. For volume V_2, we get

$$V_2 = V_1 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\kappa}}\,.$$

Under the sea level, hydrostatic pressure $p = h\varrho g$ affects the bubble, while the total pressure is the sum of hydrostatic and atmospheric pressures. We get

$$\frac{4}{3}\pi r_2^3 = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\kappa}} \frac{4}{3}\pi r_1^3\,,$$

from which we express

$$r_2 = r_1 \left(\frac{h_1\varrho g + p_\mathrm{a}}{h_2\varrho g + p_\mathrm{a}}\right)^{\frac{1}{3\kappa}} \doteq 0.85\,\mathrm{cm}\,.$$

The radius of the bubble in the depth 5 m under the sea level is 0.85 cm.

Daniela Pittnerová
daniela@fykos.cz

Problem BG ... doubly dioptric

We have a thin lens made of flint glass. We manufactured it to have optical power exactly $\varphi = 1.000\,\mathrm{D}$ for the red light. Unfortunately, flint glass has a disadvantage of relatively high dispersion. What optical power does the lens have for blue light? The refractive index of our lens is $n_\mathrm{r} = 1.628$ for the red light and $n_\mathrm{b} = 1.647$ for the blue light.

Karel was wondering about chromatic aberration.

The optical power of a thin lens satisfies the formula

$$\varphi = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (n-1)\,k\,,$$

where the constant k is the difference of multiplicative inverses of curvature radii R_1 and R_2, which remains unchanged with dispersion. Consider the following equation

$$\frac{\varphi_\mathrm{r}}{n_\mathrm{r}-1} = \frac{\varphi_\mathrm{b}}{n_\mathrm{b}-1}\,.$$

From it, we can express the optical power for blue light

$$\varphi_\mathrm{b} = \frac{n_\mathrm{b}-1}{n_\mathrm{r}-1}\varphi_\mathrm{r} = 1.030\,.$$

This calculation would not be suitable for a thick lens due to influence of the refractive index on the shift experienced by a light beam travelling across the lens.

Jáchym Bártík
tuaki@fykos.cz

Problem BH ... a molecular rotator

A diatomic molecule of oxygen with a total mass $m = 5.30\cdot 10^{-26}\,\mathrm{kg}$ is rotating around its centre of mass. The bond between the two atoms is elastic with a force constant $k = 180\,\mathrm{N{\cdot}m^{-1}}$ and length $l = 1.21\,\mathrm{A}$. What is the relative change in the length of the bond (the total change in the length divided by the original length) if the molecule starts rotating with an angular speed $\omega = 6.00\cdot 10^{12}\,\mathrm{rad{\cdot}s^{-1}}$? Assume that the atoms are point masses at the ends of the bond.

Danka remembered an exam from Physics 4.

While rotating, the elastic force F_p and centrifugal force F_c are in equilibrium. Then

$$F_\mathrm{p} = k\Delta l\,,$$
$$F_\mathrm{c} = \frac{m}{2}\omega^2\frac{l+\Delta l}{2}\,.$$

Since these forces are equal, we find

$$\frac{\Delta l}{l} = \left(\frac{4k}{m\omega^2} - 1\right)^{-1} \doteq 2.66\cdot 10^{-3}\,.$$

The relative change in length is $2.66\cdot 10^{-3}$.

Daniela Pittnerová
daniela@fykos.cz

Problem CA ... danger in class

During Social Sciences lessons, Daniel noticed that most of the neon lamps in the classroom are quite old. These lamps are $l = 1.5\,\mathrm{m}$ long and firmly attached at the endpoints. Sometimes, one of these attachments gets broken and the lamp can spin around the other end. What is the velocity of the free end of the lamp at the lowest point of its trajectory if we assume that there is no resistance? *Daniel is wondering how dangerous school is.*

To solve this problem, we will work with energy. Let's assume that a lamp attached at both ends has zero potential energy. During the fall, its centre of mass is moving down and this potential energy is changing to kinetic rotational energy. We can write

$$-\Delta E_p = \Delta E_k \,,$$
$$mg\frac{l}{2} = \frac{1}{2}I\omega^2 \,,$$

where m is the mass of the lamp, $I = \frac{1}{3}ml^2$ is the moment of inertia of a thin rod (which is an approximation for a lamp) spinning around its endpoint and ω is the angular velocity of the lamp at the lowest point. Now, we express the angular welocity as

$$\omega = \sqrt{\frac{3g}{l}} \,.$$

We can express the velocity of the free end as $v = \omega l = \sqrt{3gl} \doteq 6.6\,\mathrm{m{\cdot}s^{-1}}$.

We can also solve the problem using forces, by considering the force of gravity which acts downwards in the centre of mass (in the middle of the rod). Then, we get the same result by integration with respect to the angle between the force of gravity and the rod.

Daniela Pittnerová
daniela@fykos.cz

Problem CB ... make a wish

Danka owns a well with a uniform circular cross-section. The well is magical and makes dreams come true. Danka imagined sunny beaches of the South Pacific and threw a coin into the well. The coin elastically bounced off the walls of the well a few times. The height difference between the first and second points of impact with the wall of the well was $d_1 = 14.6\,\mathrm{m}$. The height difference between the second and third points of impact was $d_2 = 23.7\,\mathrm{m}$. Even before the coin bounced off the wall for the fourth time, Danka knew what the height difference between the third and fourth points of impact was going to be. You should calculate it as well. Assume that the coin is a point mass. *Jáchym is looking forward to holidays.*

The horizontal component v_x of the velocity is constant. The horizontal distance which the coin has to travel is always the same because the well is circular. All the impacts are elastic, so the mechanical energy is conserved and the collisions obey the law of reflection. The time between subsequent impacts is T. Let's say that the first impact occurs at the time $t = 0$ and depth $h = 0$ and the vertical component of the velocity is $v = v_0$ at that moment. Then, the formula for the depth of the i-th collision is

$$h_i = v_0 t_i + \frac{1}{2}gt_i^2 \,,$$

where $t_i = iT$. Let's denote $d_3 = h_4 - h_3$, where

$$h_i = \sum_{j=1}^{i} d_j\,.$$

We can express the vertical component of the velocity from the first equation

$$Tv_0 = d_1 - \frac{1}{2}gT^2\,,$$

and when we substitute it into the second equation, we are able to calculate the time T

$$gT^2 = d_2 - d_1\,.$$

By substituting into the third equation, we find

$$d_3 = 3v_0T + \frac{9}{2}gT^2 - d_1 - d_2 = 2d_2 - d_1 = 32.8\,\mathrm{m}\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem CC ... a paintbrush is falling

Find the time t that it takes a paintbrush to fall from the top of a roof to the ground. The roof touches the ground, its height is h, its slope is α and the coefficient of friction between the brush and the roof is f. Assume that the initial velocity of the paintbrush is zero and the slope of the roof is constant. Also, find the conditions that need to be satisfied for the brush to actually fall to the ground. *Dodo was painting his roof.*

In the direction parallel to the plane of the roof, the paintbrush (with a mass m) is affected by the parallel component of its weight, with magnitude

$$F = mg\sin\alpha\,.$$

The magnitude of the friction force F_t, acting in the opposite direction, is

$$F_\mathrm{t} = fF_\mathrm{n} = fmg\cos\alpha\,.$$

From Newton's second law, we obtain the acceleration of the brush as

$$a = \frac{F - F_\mathrm{t}}{m} = g\sin\alpha - fg\cos\alpha\,,$$

where the inequality $f < \tan\alpha$ must be satisfied - otherwise, the brush is stopped by friction. If we modify the equation for uniformly accelerated motion

$$s = \frac{1}{2}at^2$$

with $s = h/\sin\alpha$, express the time t and substitute for acceleration, we obtain

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2h}{g\left(\sin\alpha - f\cos\alpha\right)\sin\alpha}}\,.$$

Just out of interest, if we define $\sqrt{2h/g} = T$ (the time of free fall from the roof), we get

$$t = \frac{T}{\sqrt{(\sin\alpha - f\cos\alpha)\sin\alpha}}\,,$$

which gives us t as a function of the coefficient of friction and the slope of the roof. It can be seen that for $\alpha = 90^\circ$, we get free fall, and for less steep roofs, the fall slows down.

Jozef Lipták
`liptak.j@fykos.cz`

Problem CD ... sprinting on a wheelchair

Imagine that we are sitting in an electric wheelchair on an oval athletic track with length $l = 400.0\,\mathrm{m}$. Calculate the shortest time we need to drive around the oval if the inertial acceleration must not exceed $a = 0.1g$ at any time. We start in the curved part of the oval with an arbitrary non-zero velocity (which we also aim to optimise). Both linear parts and both curved parts have lengths $l/4$. *Dodo and his passion for sprinting...*

The radius of each curved part is $r = \frac{l}{4\pi}$. We can travel in these parts only with some maximal speed v_0, at which the centrifugal acceleration is exactly $a = 0.1g$. The time T_z it takes to travel through the curves is then

$$a = 0,1g = \frac{v_0^2}{r} = \frac{v_0^2 4\pi}{l}\,,$$

$$v_0 = \sqrt{\frac{lg}{40\pi}}\,,$$

$$T_\mathrm{z} = \frac{l}{4v_0} = \sqrt{\frac{5\pi l}{2g}}\,.$$

When travelling through the linear parts, the wheelchair accelerates half of the distance and decelerates the other half of the distance, with acceleration of magnitude $a = g/10$. It takes time T_p to travel the distance $l/8$ from the end of a curved part to the middle of the next linear part (or similarly from the middle of a linear part to the start of the next curved part). The equations for uniformly accelerated motion say

$$\frac{l}{8} = v_0 T_\mathrm{p} + \frac{1}{2} a T_\mathrm{p}^2 = \sqrt{\frac{lg}{40\pi}} T_\mathrm{p} + \frac{1}{20} g T_\mathrm{p}^2\,.$$

We got a quadratic equation for the time T_p. Only the positive solution is right, so

$$T_\mathrm{p} = \sqrt{\frac{5}{2\pi}}\sqrt{\frac{l}{g}}\left(\sqrt{1+\pi} - 1\right)\,.$$

The overall time it takes to drive around the oval track is therefore

$$T = 2T_\mathrm{z} + 4T_\mathrm{p} = \sqrt{\frac{l}{g}}\left(2\sqrt{\frac{5\pi}{2}} + 4\sqrt{\frac{5}{2\pi}}\left(\sqrt{1+\pi} - 1\right)\right) \approx 9.30\sqrt{\frac{l}{g}}\,.$$

After evaluating it numerically, we get $T = 59\,\mathrm{s}$, which is slower than the world records in the 400 m dash for both genders.

Jozef Lipták
liptak.j@fykos.cz

Problem CE ... dancing

Even thought it sounds quite unlikely, some Matfyz students like dancing. Daniel is trying to learn advanced steps, for example a simple pirouette. He grabs the hips of his partner and starts to spin around with her. What is the centrifugal force acting on Daniel's partner? Imagine that Daniel is really strong, so he manages to carry his partner just above the floor at the distance of his stretched arms – let's consider it $r = 0.90\,\mathrm{m}$. Assume that the spinning couple makes $f = 0.75$ spins per second, his partner weighs $m_1 = 50\,\mathrm{kg}$ and Daniel weighs $m_2 = 70\,\mathrm{kg}$.

Daniel was dreaming about other forms of procrastination.

Daniel's partner Danka is spinning with an angular velocity $\omega = 0,75 \cdot 2\pi s^{-1} = 1.5\pi s^{-1}$. Daniel and Danka are spinning around their common centre of mass, so Danka is at the distance $R = \frac{m_2 r}{m_1 + m_2} = 0.525\,\mathrm{m}$ from the axis of rotation. We can calculate the centrifugal force as

$$F = m_1 a\,,$$

where a is the centrifugal acceleration, which we can express as $a = \omega^2 R$. We get the centrifugal force

$$F = m_1 \omega^2 r \doteq 583\,\mathrm{N}\,.$$

What's interesting is that the centrifugal acceleration acting on Danka is approximately $12\,\mathrm{m\cdot s^{-1}}$, which is more than the acceleration due to gravity. However, Daniel failed and they fell down after a half-spin.

Daniela Pittnerová
daniela@fykos.cz

Problem CF ... heat and collisions

Two particles with momenta $m_1\mathbf{v_1}$ and $m_2\mathbf{v_2}$ (these are vector quantities) collided and merged. What was the heat released during the collision?

Jindra was playing with marbles.

Let's start with the laws of conservation of momentum

$$m_1\mathbf{v_1} + m_2\mathbf{v_2} = (m_1 + m_2)\mathbf{u} \tag{1}$$

and conservation of energy

$$\frac{1}{2}(m_1 + m_2)\left|\mathbf{u}\right|^2 + Q = \frac{1}{2}m_1\left|\mathbf{v_1}\right|^2 + \frac{1}{2}m_2\left|\mathbf{v_2}\right|^2\,. \tag{2}$$

We can express the velocity $\mathbf{u}$ from the equation (1) as

$$\mathbf{u} = \frac{m_1\mathbf{v_1} + m_2\mathbf{v_2}}{m_1 + m_2}$$

and substitute into the equation (2)

$$Q = \frac{1}{2}m_1 \left|\mathbf{v_1}\right|^2 + \frac{1}{2}m_2 \left|\mathbf{v_2}\right|^2 - \frac{1}{2}(m_1+m_2)\left|\frac{m_1\mathbf{v_1}+m_2\mathbf{v_2}}{m_1+m_2}\right|^2 .$$

We can simplify this expression to

$$Q = \frac{1}{2}m_1 \left|\mathbf{v_1}\right|^2 + \frac{1}{2}m_2 \left|\mathbf{v_2}\right|^2 - \frac{1}{2(m_1+m_2)}\left(m_1^2\left|\mathbf{v_1}\right|^2 + 2m_1m_2\mathbf{v_1}\cdot\mathbf{v_2} + m_2^2\left|\mathbf{v_2}\right|^2\right) ,$$
$$Q = \frac{m_1m_2}{2(m_1+m_2)}\left|\mathbf{v_1}\right|^2 - \frac{m_1m_2}{m_1+m_2}\mathbf{v_1}\cdot\mathbf{v_2} + \frac{m_1m_2}{2(m_1+m_2)}\left|\mathbf{v_2}\right|^2 ,$$
$$Q = \frac{m_1m_2}{2(m_1+m_2)}\left(\mathbf{v_1}-\mathbf{v_2}\right)^2 .$$

As expected, the heat depends only on the difference of the velocities and so, it is the same in all inertial reference frames.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem CG ... cooking

Danka cooks on a hob with input energy consumption (power) P. She pours 2.00 l of water with temperature 40 °C into a stock pot with temperature 23 °C. The water starts boiling after a time $t_1 = 6\,\mathrm{min}$. Danka then empties the pot and lets it cool to 70 °C. At this moment, Danka pours 2.00 l of water with temperature 40 °C into the pot again. How much time does she save if she wants to wait until the water starts boiling again? The stock pot has temperature 105 °C when the water is boiling. The hob heats the pot and water with efficiency $\eta = 0.85$. The heat capacity of the pot is $C = 439\,\mathrm{J\cdot K^{-1}}$ and the specific heat capacity of water is $c_\mathrm{v} = 4\,180\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$. Danka uses the electric cooker at the dormitory.

Let's denote the important temperatures by $T_1 = 23\,^\circ\mathrm{C}$, $T_2 = 40\,^\circ\mathrm{C}$, $T_3 = 70\,^\circ\mathrm{C}$, $T_4 = 105\,^\circ\mathrm{C}$. Water boils at the temperature $T_v = 100\,^\circ\mathrm{C}$. The heat transfer between the hob and the pot with the water is described by the calorimetry equation

$$\eta P t_1 = C\left(T_4 - T_1\right) + c_\mathrm{v} V \varrho \left(T_\mathrm{v} - T_2\right) ,$$

where V is the volume of the water and ϱ is its density. The heat transfer during the second heating process is described by the equation

$$\eta P t_2 = C\left(T_4 - T_3\right) + c_\mathrm{v} V \varrho \left(T_\mathrm{v} - T_2\right) .$$

We can eliminate P and η and write

$$w = \frac{t_2}{t_1} = \frac{C\left(T_4 - T_3\right) + c_\mathrm{v} V \varrho \left(T_\mathrm{v} - T_2\right)}{C\left(T_4 - T_1\right) + c_\mathrm{v} V \varrho \left(T_\mathrm{v} - T_2\right)} .$$

The ratio is $w = 0.961\,6$. We can calculate the time difference

$$\Delta t = (w-1)\,t_1 \doteq -14\,\mathrm{s}.$$

The negative sign means that the water starts boiling 14 s earlier than in the first case. Danka hasn't saved much time. In reality, the heat capacity of the hob is more important than the heat capacity of the pot. When the hob is hot, the water would boil significantly sooner, in fact.

Daniela Pittnerová
daniela@fykos.cz

Problem CH ... water in an angular tube

We have a tube with water flowing through it, with a mass flow rate Q and a velocity v. At one point, this tube is bent in such a way that it has two arms with an angle α between them (so $\alpha = \pi$ means that there is no bend). What is the magnitute of the force the water is exerting on this bend? *Legolas is glad that he has no water in his knee.*

Through the angular tube, water with mass $\mathrm{d}m = Q\mathrm{d}t$ flows during a small time period $\mathrm{d}t$. This changes its velocity by $\mathbf{u} = \mathbf{v_2} - \mathbf{v_1}$. From Newton's first and third law, we get the force the water exerts on the angular tube

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}m\mathbf{u}}{\mathrm{d}t} = Q\mathbf{u}\,.$$

After applying some geometry, we get

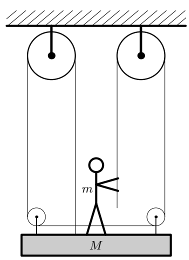
$$F = 2Qv\cos\left(\frac{\alpha}{2}\right)\,.$$

For $\alpha = \pi$, we get $F_\mathrm{v} = 0$, which corresponds to the fact that in such a case, the water would be unaffected.

Šimon Pajger
legolas@fykos.cz

Problem DA ... the floor is lava

A member of FYKOS with mass $m = 50\,\mathrm{kg}$ (depicted in the figure) is pulling a rope with a constant force F. The mass of the platform is $M = 50\,\mathrm{kg}$. Can the FYKOS member lift himself above the platform? If yes, what force does he have to pull with to do so? Neglect the moments of inertia of all pulleys. *Matěj fell into a pit and couldn't get out.*



An important property of every rope in a system of ideal pulleys is that the force of tension in it is everywhere the same. The platform is pulled only by the rope, with an acceleration

$$a_\mathrm{p} = \frac{3F}{M}\,,$$

because there are three parts of the rope and each of them is exerting a force F on the platform. The FYKOS member is pulled upwards by the rope with an acceleration

$$a_\mathrm{F} = \frac{F}{m}\,,$$

since the FYKOS member is pulling only one rope and the force it exerts on him is F. Since after substitution, $a_\mathrm{F} < a_\mathrm{p}$ holds for every positive force F, the upwards acceleration of the platform will always be greater than that of the FYKOS member. Therefore, he can never pull himself above the platform.

Accounting for gravity does not affect the result because g can be subtracted from both accelerations and the inequality remains unchanged.

Matěj Mezera
m.mezera@fykos.cz

Problem DB ... lift mechanics

A container with water is placed inside a stationary lift. In the container, there is also a cuboidal weight made of aluminium (with dimensions $x = 3.00\,\mathrm{cm}$, $y = 4.00\,\mathrm{cm}$ and $z = 5.00\,\mathrm{cm}$), which is fully submerged in the water. The weight is hanging on a massless spring with a spring constant $k = 230\,\mathrm{N{\cdot}m^{-1}}$, which is attached to the ceiling and initially stretched. The lift begins to move upwards with a constant acceleration $a = 3.00\,\mathrm{m{\cdot}s^{-2}}$. Find the ratio of the spring's elongation when the lift is moving to its initial elongation. The density of aluminium is $\varrho_{\mathrm{Al}} = 2\,700\,\mathrm{kg{\cdot}m^{-3}}$. Assume that the spring doesn't stretch far enough to reach the floor.

Dodo carried a plate of soup away from Danka.

In the case when the lift is stationary, the force of gravity F_g, which is acting on the weight, is compensated by the buoyant force F_v and the tensile force F_p of the spring. It satisfies the force balance equation

$$V\varrho_\mathrm{h} g = V\varrho_\mathrm{v} g + k\delta l_0\,,$$

where $V = xyz$ is the volume of the weight, ϱ_v is the density of water and δl_0 is the initial elongation of the spring. The accelerating lift is indistinguishable from a stationary one which is influenced by gravity $a + g$. Therefore, the force balance is

$$V\varrho_\mathrm{h}\,(a+g) = V\varrho_\mathrm{v}\,(a+g) + k\delta l\,.$$

Now we simply express the elongations of the spring from both equations and calculate their ratio as

$$\frac{\delta l}{\delta l_0} = 1 + \frac{a}{g} \doteq 1.31\,.$$

The elongation of the spring in the accelerating lift is 1.31 times larger.

Daniela Pittnerová
daniela@fykos.cz

Problem DC ... rotation of the velocity vector

Certainly, you have already calculated common projectile motion - for instance, the trajectory of a ball which was kicked under an initial angle α (with respect to the ground) and with an initial speed v_0. Find out how the magnitude of angular velocity ω of its velocity vector $\mathbf{v}$ depends on the initial speed v_0, the initial angle α, the instantaneous speed v and the gravitational acceleration g.

Robo was thinking during a PE class.

The acceleration acting in the direction perpendicular to the trajectory (the radial direction) can be expressed as $a_\mathrm{n} = g\cos\vartheta$, where ϑ is angle between the velocity vector and the horizontal direction. Since the instantaneous velocity vector is tangential to the trajectory, the angle satisfies $\cos\vartheta = v_x/v$. No force acts on the ball in the horizontal direction, therefore its horizontal velocity remains constant, $v_x = v_0\cos\alpha$. We also know that the centripetal acceleration satisfies

$$a_\mathrm{c} = \frac{v^2}{R} = \omega v\,,$$

where R is the radius of curvature of the trajectory in the current position. The motion of the ball may be locally approximated by circular motion. For this motion, the angular velocity of the velocity vector is the same as the angular velocity of rotation of the body around the centre of curvature. We write the equation for the accelerations and express the desired angular velocity

$$\omega = \frac{a_\mathrm{c}}{v} = \frac{a_\mathrm{n}}{v} = \frac{g v_0\cos\alpha}{v^2}\,.$$

Róbert Jurčo
`robert.jurco@fykos.cz`

Problem DD ... heavy can

Imagine a symmetric can. Its mass is m, its height is H and the area of each base is S. There is a liquid with density ϱ in the can.

We want the height of the centre of mass of the system can+liquid to be the smallest possible. For what height h of the liquid does it happen? *Lego loves beer and physics.*

The can is symmetric, so its centre of mass is $H/2$ above the ground. The height of the centre of mass of the liquid is similarly $h/2$. The height of their common centre of mass is

$$x = \frac{m_1x_1 + m_2x_2}{m_1+m_2} = \frac{mH/2 + Sh\varrho h/2}{m + Sh\varrho}\,.$$

One of the ways to solve this is to differentiate with respect to h and find local extrema. However, there's a quicker (and more elegant) way. The centre of mass is in its lowest position when it is at the same height as the level of the liquid. If we pour more water in such a situation, the centre of mass obviously rises up. If we slop some liquid (which is equivalent to placing a liquid with density $-\varrho$ below the current level of the liquid), the centre of mass rises again. We can therefore conclude that the minimum height of the centre of mass is $x = h$,

$$h = \frac{1}{2}\frac{mH + Sh^2\varrho}{m + Sh\varrho}\,,$$
$$0 = S\varrho h^2 + 2mh - mH\,.$$

The solution is the only positive root of this equation

$$h = \frac{-2m + \sqrt{4m^2 + 4Hms\varrho}}{2S\varrho} = \frac{\sqrt{m^2 + HmS\varrho} - m}{S\varrho}\,.$$

Šimon Pajger
`legolas@fykos.cz`

Problem DE ... Jindra the tractor driver

Tractor driver Jindra has an uncommon hobby – he likes moving from some point A to B in the shortest possible time. Now he finds himself in a field, at the point A, and the point B is located at the distance d to the east from A. The speed of the tractor depends on the azimuth of its motion as $v = v_0 \left|\cos\alpha\right|$ (so $v = v_0$ if it's moving northwards or southwards). How long does it take Jindra to reach point B? *Jindra still hasn't acquired a driving licence.*

We solve the problem using a trick. Imagine a Cartesian coordinate system in the field such that its x axis points to the east and the y axis points to the north. Jindra has to move by a distance d in the direction of the x axis. The azimuth is measured clockwise from the northward direction. The x-component of the tractor's velocity can be calculated by multiplying the speed by the sine of the azimuth

$$v_x = v_0 \left|\cos\alpha\right| \sin\alpha\,.$$

We are interested in motion in the positive x-direction, i.e. $\alpha \in \langle 0^\circ, 180^\circ\rangle$. Consider that

$$v_x = v_0 \left|\cos\alpha \sin\alpha\right|\,.$$

We find the azimuth corresponding to the maximal velocity by placing the derivative equal to zero. Since we know that in the given range, the velocity is always positive, it's enough to maximise the expression inside the absolute value

$$0 = \frac{\mathrm{d}\cos\alpha\sin\alpha}{\mathrm{d}\alpha} = \cos^2\alpha - \sin^2\alpha\,.$$

The solutions are $\alpha_1 = 45^\circ$ and $\alpha_2 = 135^\circ$. Alternatively, we recall double angle formula $\left|\cos\alpha\sin\alpha\right| = \left|\sin 2\alpha\right|/2$, so we get maxima of 1 for $2\alpha_1 = 90^\circ$ and $2\alpha_2 = 270^\circ$.

Now we are asking: "Can Jindra move in such a direction that his velocity in the positive direction of the x axis would always be equal to this maximum?" Yes, he can. For example, if he moves on the azimuth $\alpha_1 = 45^\circ$ half of the route and on the azimuth $\alpha_2 = 135^\circ$ afterwards. This ensures that his final y coordinate remains the same as at the beginning. Assuming that he always moves with maximal velocity in the direction of the x axis, there is no faster route.

We can calculate the time necessary to move to point B as

$$t = \frac{d}{v_x} = \frac{d}{v_0\left|\cos\alpha_1 \sin\alpha_1\right|} = \frac{2d}{v_0}\,.$$

The time the route takes Jindra is $t = 2d/v_0$.

Jindřich Jelínek
jjelinek@fykos.cz

Problem DF ... lens-like mirror

We have a thin plano-convex lens with a radius of curvature R and a refractive index n. Onto the convex side of the lens, we place a shiny foil so that it behaves as a convex spherical mirror with a radius R. Find the (positive) focal length of this system.

Matěj is checking himself out in a mirror.

We will find the solution using geometrical methods. A focal point is a place into which light beams parallel to the optical axis are focused. Let us take a beam, parallel to the optical axis, at a small distance x from the axis. Since the lens is thin, we will use the paraxial approximation $\sin x \approx x$, $\tan x \approx x$.

- **passage through the flat side** - The beam doesn't change its direction because it's incident on this side perpendicularly.
- **reflection from the mirror** - The beam, parallel to the axis, is reflected towards the focal point of the mirror, at the distance $f_\mathrm{z} = \frac{R}{2}$. Therefore, the angle between the axis and the reflected beam is $\alpha = \frac{x}{f_\mathrm{z}}$.
- **second passage through the flat side** - The beam refracts in such a way that after passing through the lens, its angle with respect to the axis is β, which is given by Snell's law $n\alpha = \beta$.

Since the lens is thin, the beam exits the lens again at the distance x from the optical axis. The difference is that now, it's at the angle β with respect to it. The focal point we're looking for is the place at which it crosses the optical axis and its distance from the lens is

$$f = \frac{x}{\beta} = \frac{x}{n\alpha} = \frac{x}{n\frac{x}{f_\mathrm{z}}} = \frac{f_\mathrm{z}}{n} = \frac{R}{2n}\,.$$

We can see that the point of intersection doesn't depend on the distance x between the beam and the optical axis and therefore, all beams intersect the axis at the same point. Of course, this only applies for sufficiently small x, when we can use the paraxial approximation. Otherwise, we would find that both the lens and the mirror have optical defects, causing the parallel beams further away from the axis to miss this focal point.

Matěj Mezera
`m.mezera@fykos.cz`

Problem DG ... at the absolute bottom

Dano has a dry well with constant circular cross-section and depth $h = 23.0\,\mathrm{m}$. When he watches the sky from the centre at the bottom of the well, he sees all the stars which have a smaller zenith distance than $\alpha = 8.00^\circ$. What is the minimum volume of an unknown liquid with a refractive index $n = 2.31$ that Dano has to pour into the well in order to see all the stars with the zenith distance 2α? *... however, Jáchym knows how it ends.*

In both cases, Dano's view is limited by the edge of the well. We can describe the initial situation by the equation $r = h\tan\alpha$, where r is the radius of the well. After the liquid is poured into the well, refraction of light occurs at its surface. The angle of refraction is 2α and we can denote the angle of incidence by β. From Snell's law, we get

$$\sin 2\alpha = n\sin\beta\,,$$

where n is the refractive index of the liquid. Let h_1 be the distance from the bottom to the surface of the liquid and h_2 the distance between the surface and the upper edge of the well, so $h_1 + h_2 = h$. Similarly, let r_1 be the distance of the centre from the point of refraction of a furthest ray, measured along the bottom of the well, and r_2 the distance of the point of refraction from the edge of the well, so $r_1 + r_2 = r$. Now, we can write the equation for the upper triangle

$$\sin 2\alpha = \frac{r_2}{\sqrt{r_2^2 + h_2^2}}\,,$$

from which we get

$$r_2 = \left(\sin^{-2}(2\alpha) - 1\right)^{-\frac{1}{2}} h_2 = \tan(2\alpha)\, h_2 = k_2 h_2\,.$$

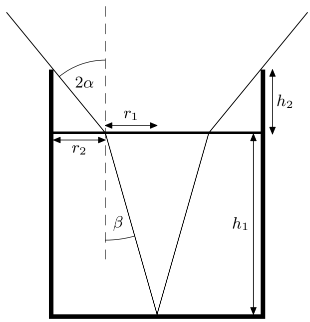
Similarly, for the lower triangle, we can write the equation

$$r_1 = \left(\sin^{-2}\beta - 1\right)^{-\frac{1}{2}} h_1 = \left(n^2 \sin^{-2}(2\alpha) - 1\right)^{-\frac{1}{2}} h_1 = k_1 h_1\,.$$

In these equations, we have used the sine in order to directly use Snell's law. We hid the ugly expressions into the constants k_1 and k_2. Now, we substitute for r_1 and r_2 in the equation $r = r_1 + r_2$, which gives us a system of two equations for two variables

$$h = h_1 + h_2\,,$$
$$r = k_1 h_1 + k_2 h_2\,.$$

Then, the solution for the height h is



$$h_1 = \frac{k_2 h - r}{k_2 - k_1}\,.$$

However, we want to know the total volume of the liquid, so we have to multiply the height by the area of the cross-section πr^2. Now, we can substitute for r and k_2 and get the final result

$$V = \pi r^2 h_1 = \pi h^3 \frac{\tan 2\alpha - \tan\alpha}{\tan 2\alpha - k_1}\tan^2\alpha \doteq 663\,\mathrm{m}^3\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem DH ... retroreflection

Retroreflective elements are called such because they reflect light back to the source. There are two ways of constructing them. We can cover the surface by miniature corner reflectors (like in bike reflectors) or we can cover it by a reflective material and partially embed transparent balls into it (like in white strips on a reflective vest). Calculate an ideal refractive index of the balls in order to reflect back most of the incoming light.

Jindra was solving the competition N-trophy[9]*.*

The material is a retroreflector when the incoming ray is parallel to the outgoing ray. Let's draw a line connecting the source of the light and the centre of a ball and assume that the diameter of the ball is insignificantly small. An incoming ray is parallel to the optical axis, it impacts the surface of the ball at a perpendicular distance h from the optical axis and it forms an angle α with the normal at the point of impact. The radius of the ball is R. The paraxial approximation is valid while $h/R \ll 1$. Rays that are further away from the axis are reflected more to the sides, but in this case, such divergence of the reflected light isn't too significant. We can write

$$\sin\alpha = \frac{h}{R} \approx \alpha\,.$$

From Snell's law, we calculate the angle of refraction β, and we also assume $\beta \ll 1$

$$\sin\alpha = n\sin\beta\,,$$

$$\frac{h}{R} = n\beta\,.$$

Retroreflection occurs when the refracted ray hits the point where the optical axis intersects the opposite surface of the ball. Then, the two rays are symmetrical with respect to the optical axis and the outgoing ray is parallel to the optical axis (like the incoming ray). Under the small angle approximation, we can write $\beta = h/(2R)$, so

$$\frac{h}{R} = n\frac{h}{2R}\,,$$

$$n = 2\,.$$

The ideal refractive index of the ball is $n = 2$.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem EA ... jerked around

Find the mean square modulus of the change in the vector of velocity of a pollen particle with mass $M = 250\,\mathrm{ng}$ *when this particle collides head on with an atom of argon in air (i.e. in such a way that they are moving directly towards each other). The temperature is* $t = 25.3\,\mathrm{^\circ C}$ *and the pressure is* $p = 1\,003\,\mathrm{hPa}$.

Dodo took a full metro back from school.

A perfectly elastic collision implies conservation of both momentum and kinetic energy. If the pollen particle remains at rest at the beginning (we can simply choose a coordinate system where it does), then after the collision, it is moving with a velocity u which obeys

$$mv = Mu + mv'\,,$$

$$mv^2 = Mu^2 + mv'^2\,,$$

where m is the mass of an atom of argon and v, v' are the initial and new velocity of the atom respectively. After substituting for v' from the first equation, we obtain

$$u = \frac{2vm}{M+m} \approx \frac{2vm}{M} \,.$$

We're supposed to find the mean square value of this quantity. From the equipartition theorem, we have

$$\frac{1}{2} m v_\mathrm{k}^2 = \frac{s}{2} k_\mathrm{B} T \,,$$

where T is the thermodynamic temperature of the gas and s is the number of active degrees of freedom. After substitution, we get the mean square value of u as

$$u_\mathrm{k} = \frac{2\sqrt{s k_\mathrm{B} T m}}{M} \,.$$

The mass of an argon atom can be calculated using the Avogadro constant and molar mass as $m = M_\mathrm{m}/N_\mathrm{A} = 6.64 \cdot 10^{-26}\,\mathrm{kg}$. An argon atom has only translational degrees of freedom, so $s = 3$. The numeric value is $u_\mathrm{k} = 2.3 \cdot 10^{-13}\,\mathrm{m{\cdot}s^{-1}}$.

Jozef Lipták
liptak.j@fykos.cz

Problem EB ... green shift

Chuck Norris has such a fast car that he sometimes drives through an orange traffic light because he sees it as green. But that's nothing. Paťo (the one who ran a 12-minute run in 6 minutes) runs so fast that he sees red as green. The wavelengths of green, orange and red light are $\lambda_\mathrm{z} = 550\mathrm{nm}$, $\lambda_\mathrm{o} = 600\mathrm{nm}$ and $\lambda_\mathrm{c} = 700\mathrm{nm}$ respectively. What is the difference between the speeds of Paťo and Chuck Norris's car? *Lego was running towards a crosswalk.*

We may assume that Paťo's speed and Chuck Norris's car's speed are both relativistic. At the same time, the waves are light waves, not sound waves, so a relativistic version of the Doppler Law must be used

$$\lambda_\mathrm{observer} = \lambda_\mathrm{source} \sqrt{\frac{1 - v/c}{1 + v/c}} \,.$$

This formula can either be found in the tables or it can be derived from the original Doppler Law by considering relativistic effects. The ratio v/c is commonly referred to as β. In addition, if we denote $\lambda_\mathrm{observer}/\lambda_\mathrm{source} = \alpha$, we can express

$$\beta = \frac{1 - \alpha^2}{1 + \alpha^2} \,.$$

Now we get the speed of Chuck Norris's car

$$v_\mathrm{Chuck} = \frac{1 - (\lambda_\mathrm{z}/\lambda_\mathrm{o})^2}{1 + (\lambda_\mathrm{z}/\lambda_\mathrm{o})^2} c \doteq 2.60 \cdot 10^7\,\mathrm{m{\cdot}s^{-1}} \,.$$

Similarly, we have for Paťo

$$v_\mathrm{Pato} = \frac{1 - (\lambda_\mathrm{z}/\lambda_\mathrm{c})^2}{1 + (\lambda_\mathrm{z}/\lambda_\mathrm{c})^2} c \doteq 7.10 \cdot 10^7\,\mathrm{m{\cdot}s^{-1}} \,.$$

Therefore, Paťo is faster than Chuck Norris's car by

$$v_{\text{Pato}} - v_{\text{Chuck}} = \frac{2\lambda_{\text{z}}^2\left(\lambda_{\text{c}}^2 - \lambda_{\text{o}}^2\right)}{\left(\lambda_{\text{c}}^2 + \lambda_{\text{z}}^2\right)\left(\lambda_{\text{o}}^2 + \lambda_{\text{z}}^2\right)} c \doteq 0.15c \doteq 4.49 \cdot 10^7\,\mathrm{m{\cdot}s^{-1}}$$

Šimon Pajger
legolas@fykos.cz

Problem EC ... Are we climbing?

We are sitting in a train inside a tunnel. The train driver decided to let the train move only by its own inertia. What is the angle α between the water level in a glass and the floor of the train? Assume that when the train is standing on a horizontal surface, the floor is also horizontal. The elevation of the track is $t = 1.2\,\%$ and the rolling resistance coefficient between the track and the wheels of the train is $f = 0.002$. *Dodo was travelling back to school.*

The train is moving uphill. First, let's convert the elevation from percent to an angle. Elevation is the change in altitude per unit horizontal displacement, so the angle is $\varphi = \arctan t$. The motion of the train is directly decelerated by the tangential component of the gravity of Earth and by the friction given by its normal component. The acceleration of the train in the direction of motion is

$$a_{\text{v}} = a_{\text{t}} + f a_{\text{n}} = g\left(\sin\varphi + f\cos\varphi\right)\,.$$

In the non-inertial reference frame connected with the train, there is the inertial acceleration parallel with the floor, with magnitude a_{v}, but in the opposite direction (in the direction of motion of the train), and the real acceleration - the acceleration due to gravity, with magnitude g and at an angle φ from the normal to the floor, towards the back of the train. To find the angle between the water level and the floor, we need to find the angle between the vertical and the net force because these angles are the same. Decomposing the acceleration due to gravity into the normal and tangential component to the train floor, we get

$$a_{\text{k}} = g\cos\varphi\,,$$
$$a_{\text{r}} = g\sin\varphi\,.$$

Adding the inertial acceleration $a_{\text{z}} = -a_{\text{v}}$ to this tangential component, we get the desired angle as the arctangent of the ratio of magnitudes of accelerations in the “horizontal” and the “vertical” direction

$$\alpha = \arctan\frac{a_{\text{r}} + a_{\text{z}}}{a_{\text{k}}} = \arctan\frac{a_{\text{r}} - a_{\text{v}}}{a_{\text{k}}} = \arctan\frac{\sin\varphi - \left(\sin\varphi + f\cos\varphi\right)}{\cos\varphi} = -\arctan f \doteq -0.11\,^\circ\,.$$

We are interested only in its magnitude, the sign gives information about the orientation of the water level. The answer is $\alpha \doteq 0.11\,^\circ$. An interesting fact that we can notice from this solution is that we cannot say whether the train goes uphill or not. This follows from the Einstein equivalence principle.

Jozef Lipták
liptak.j@fykos.cz

Problem ED ... hole in a planet

At a large distance from the Earth, there is a rocket which has run out of fuel. The rocket is attracted to the Earth and NASA scientists are trying to figure out how to save the crew. Honza suggests digging a hole through the Earth, so that the rocket would not hit the ground, but fly through the tunnel. For simplicity, assume that the rocket is initially at rest at an infinite distance from the Earth. The tunnel through the Earth is aligned with the direction of the fall of the rocket. Honza wants to know what the velocity of the rocket would be in the middle of the Earth. Assume that the Earth is homogeneous. *Robo found himself inside a planet.*

The mass of the Earth is M and its radius is R. At the beginning, the rocket has both zero potential energy and zero kinetic energy, and the mechanical energy is conserved, so we know that in the centre of the Earth, the sum of the kinetic and potential energies would be zero again.

$$0 = \frac{1}{2}mv^2 + E_p\,.$$

Now, we need to find the gravitational potential in the centre of the Earth φ. From the potential, one can easily find the potential energy as

$$E_p = m\varphi\,.$$

We know that the gravitational potential of the Earth is given by

$$\varphi_R = -\frac{GM}{R}\,.$$

From Gauss's law for gravity, we know the intensity of the gravitational field inside the Earth at a distance r from the centre

$$E = -\frac{GM_r}{r^2}\,,$$

where M_r is the mass below the radius r. We assumed that the Earth is homogeneous, so the mass M_r is directly proportional to the volume, which is proportional to the cube of the radius

$$M_r = \frac{r^3}{R^3}M\,.$$

The potential in the centre of the Earth φ_0 is the sum of the potential on the surface and the integral of $-E$ from the surface to the centre of the Earth

$$\varphi_0 = -\frac{3GM}{2R}\,.$$

Substituting for E_p in the law of energy conservation, we get

$$0 = \frac{1}{2}mv^2 - \frac{3GMm}{2R}$$

This gives the velocity of the rocket in the centre of the Earth $v = \sqrt{\frac{3GM}{R}} = 13.7 \cdot 10^3\ \mathrm{m{\cdot}s^{-1}}$.

Róbert Jurčo
robert.jurco@fykos.cz

Problem EE ... slippery rope

Dodo has a well and a bucket. The mass of the bucket is $m = 1.75\,\mathrm{kg}$ and it can hold at most $V = 15.2\,\mathrm{l}$ of water. The bucket is hanging from a rope, which is thrown over a fixed log with a circular cross-section. When Dodo pulls the bucket full of water upwards, he pulls the rope with a force $F = 237\,\mathrm{N}$. Find the force which Dodo must exert in order to drop the empty bucket down with a constant velocity. *Jáchym prefers aid climbing.*

In this problem, it is essential to know how friction acts on a rope attached around a cylinder with circular cross-section. The exact answer to that can be found e.g. in the solutions of FYKOS problems **27.III.5** and **32.VI.4**. In this case, it is only necessary to know that the ratio between the force exerted on one end of the rope and the force exerted on the other end is something similar to an exponential of an expression involving the coefficient of friction and the total angle of contact between the rope and cylinder. All these values are the same in both situations, so the ratio of the forces on both ends of the rope is also the same.

In the first case, the gravity of the bucket with water is

$$F_\mathrm{s} = (m + \varrho V)\, g\,,$$

and it's the "weaker" force, while on the other end, Dodo pulls with a force F. In the second case, the force F' which Dodo utilises to slow down the rope is the "weaker" force. The force that acts at the other end is the gravity of the empty bucket itself

$$F_\mathrm{b} = mg\,.$$

From the observation above, we get

$$\frac{F}{F_\mathrm{s}} = \frac{F_\mathrm{b}}{F'} = \mathrm{const} > 1\,.$$

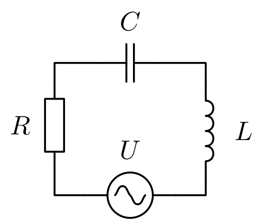
From there, we can simply express the desired force

$$F' = \frac{F_\mathrm{b} F_\mathrm{s}}{F} = \frac{m + \varrho V}{F} m g^2 \doteq 12.0\,\mathrm{N}\,.$$

Finally, we can see that it really is less than the approximately $17\,\mathrm{N}$ required to balance an empty bucket without friction.

Jáchym Bártík
`tuaki@fykos.cz`

Problem EF ... RLC for sure this time



Lego took Dodo's circuit from the Online Physics Brawl and slightly changed it. Now it is the circuit shown in the figure, composed of a coil with inductance $L = 10.0\,\mathrm{mH}$, a capacitor with capacitance $C = 4.70\,\mathrm{\mu F}$, a resistor with resistance $R = 1.00\,\mathrm{k\Omega}$, and an AC voltage source with the effective value of voltage $U_\mathrm{ef} = 230\,\mathrm{V}$ and adjustable frequency. Lego set the frequency of the AC source in such a way that the amplitude of the current would be maximised. What is the power drawn by the whole circuit?

Lego felt sorry for FOL participants... so he set a similar troll problem here as well.

We could solve the problem using complex numbers, but we can solve it more easily using some basic knowledge about alternating current.

For example, it is useful to know that for a series RLC circuit (i.e. in which the current can't bypass the resistor through the coil or capacitor), the current (for a given voltage) is maximised when the frequency of the current is equal to the resonant frequency of the given circuit. That's because at the resonant frequency, the impedances of the coil and capacitor cancel out, so $Z = R$.

Therefore, for the effective value of the current, we have $I_{\mathrm{ef}} = U_{\mathrm{ef}}/Z = U_{\mathrm{ef}}/R$. The current is not phase shifted in any way with respect to the voltage, so $\varphi = 0$.

What are the effective values of the voltage and current? Effective value is defined as the amplitude of a given quantity divided by the square root of 2 (so $I_{\mathrm{ef}} = I_{\max}/\sqrt{2}$ and accordingly for the voltage). It's defined that way so that the formula $P = U_{\mathrm{ef}} I_{\mathrm{ef}} \cos\varphi$ would hold. Since $\cos 0 = 1$, we get the power as

$$P = U_{\mathrm{ef}} I_{\mathrm{ef}} = \frac{U_{\mathrm{ef}}^2}{R} = 52.9\,\mathrm{W}\,,$$

which is approximately the power of a standard light bulb and very accurately, the solution of the already mentioned problem from the Online Physics Brawl.

Šimon Pajger
`legolas@fykos.cz`

Problem EG ... rotation of polarisation

We all know that when horizontally polarised light hits an ideal vertical polarisation filter, the light intensity behind the filter is zero - the light cannot pass through the filter. However, if we place another polarisation filter between the two, light can pass through the last filter. Suppose that we have N filters in a series such that each one is rotated by the same angle δ relative to the previous filter. Additionally, the last filter is rotated by 90° relative to the first one. We let light with an original intensity I_0 pass through the filters. For a large value of N, estimate how much the intensity of the light decreases after passing through all the filters.

Štěpán makes photons pass through.

A filter rotated by δ relative to the polarisation decreases the strength of the electric field passing through it from $\mathbf{E}_1$ to $\mathbf{E}_2$, where

$$|\mathbf{E_2}| = |\mathbf{E_1}| \cos\delta\,,$$

Since the intensity is proportional to the square of the field strength, it decreases from I_1 to I_2, where

$$I_2 = I_1 \cos^2\delta\,.$$

Therefore, after passing through N filters,

$$I = I_0 \cos^{2N}\delta = I_0 \cos^{2N}\left(\frac{\pi}{2N}\right)\,,$$

where I_0 is the initial intensity and we plugged in $\delta = \frac{\pi}{2N}$. For large values of N, we can approximate the cosine as

$$\cos\left(\frac{\pi}{2N}\right) \approx 1 - \frac{\pi^2}{2\cdot(2N)^2} = 1 - \frac{\pi^2}{8N^2}$$

and then, we can write

$$I \approx (1 - \frac{\pi^2}{8N^2})^{2N} I_0 \approx \left(1 - \frac{\pi^2}{4N}\right) I_0\,.$$

The intensity decreased by $\frac{\pi^2}{4N} I_0$.

Štěpán Marek
stepan.marek@fykos.cz

Problem EH ... TNT planet

Consider a sphere of TNT floating freely in space. Assume that firing one kilogram of TNT releases 4.184 MJ of energy, which is immediately converted into kinetic energy of the reaction products (whose mass is the same as the mass of the original TNT, 1 kg). What is the radius of the largest sphere with density $\varrho = 1\,650\,\mathrm{kg{\cdot}m^{-3}}$ that can completely scatter through the explosion (i.e. all its mass is thrown to infinity, where it does not gravitationally affect itself anymore)? *Jáchym joined Karel and decided to also destroy a planet.*

Let our planet have a radius R and a mass

$$M = \frac{4}{3}\pi R^3 \varrho\,.$$

If we denote the calorific value of trinitrotoluene as $H = 4.184\,\mathrm{MJ{\cdot}kg^{-1}}$, the total energy of the explosion is

$$E_\mathrm{v} = HM = \frac{4}{3}\pi R^3 H\varrho\,.$$

Now we take the upper layer of the planet, with width $\mathrm{d}r$, and move it to infinity. The gravitational potential on the surface of a planet with radius r and mass m is

$$\varphi = -\frac{Gm}{r}\,,$$

so we have to add energy

$$\mathrm{d}E = -\varphi \mathrm{d}m\,,$$

where the mass of the layer $\mathrm{d}m$ is calculated as

$$\mathrm{d}m = 4\pi r^2 \varrho \mathrm{d}r\,.$$

By doing so, we get rid of the upper layer of the planet, and thereby reduce its radius and weight

$$m = \frac{4}{3}\pi r^3 \varrho\,.$$

We calculate the total gravitational energy needed to move all parts of the planet to infinity as the integral

$$E_\mathrm{g} = \int_0^R \mathrm{d}E = -\int_0^R \varphi\,\mathrm{d}m = \frac{16\pi^2 G\varrho^2}{3}\int_0^R r^4\,\mathrm{d}r = \frac{16\pi^2 G\varrho^2}{3}\left[\frac{r^5}{5}\right]_0^R = \frac{16\pi^2 G R^5 \varrho^2}{15}\,.$$

We see that $E_{\,g}$ grows with the fifth power of the radius, while E_{v} only grows with the third power. Therefore, for all R larger than R_0 (for which $E_{\text{g}} > E_{\,v}$), we will not be able to disperse the planet perfectly. We can write the result as

$$R_0 = \sqrt{\frac{5H}{4\pi G\varrho}} \doteq 3\,888\,\mathrm{km}\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem FA ... Jindra II. the tractor driver

Jindra the tractor driver has an unusual hobby – he likes moving from some point A to B in the shortest possible time. At this moment, Jindra with his tractor is in a field, at the point A, and the point B is located at a distance r from the point A in the same field. At a distance d from the line segment AB, there is a driveway running parallel to it. In the field, the tractor moves with the same velocity u in all directions and on the driveway, it moves with a velocity v, where $v > u$. Jindra found out that it does not matter if he drives from the point A to the point B directly or if he first drives from the point A onto the driveway, then along the driveway and after a while, back into the field towards the point B. In both cases, the time of the journey is t. Express the ratio d/r using u and v.

Jindra wanted to experience total tractor reflection.

First, we must derive the formulas describing the time of driving directly from A to B and the time of travel when using the driveway. Let t_{p} denote the travel time for the direct case. We can calculate it from the formula for uniform linear motion

$$t_{\text{p}} = \frac{r}{u}\,.$$

It gets slightly more complicated with the driveway. Consider that there is an infinite number of ways in which the tractor can get on the driveway, as well as an infinite number of ways to get off it. We have to find the most efficient one. The tractor rides from the point A to the point where it gets on the driveway in a straight line[1]. It also rides in a straight line from the point where it leaves the road to the point B. Let's denote the angle between the trajectory of the tractor before it reaches the driveway and a normal to the driveway by α_{A}. Similarly, let's denote the angle of its trajectory after it leaves the driveway by α_{B}. The total time of the ride is

$$t_{\text{s}} = \frac{d}{u\cos\alpha_{\text{A}}} + \frac{r - d\tan\alpha_{\text{A}} - d\tan\alpha_{\text{B}}}{v} + \frac{d}{u\cos\alpha_{\text{B}}}\,. \tag{3}$$

We want to choose the angles α_{A} and α_{B} in such a way that the time t_{s} is minimal. A local extremum of a function may be found by placing the derivative equal to zero

$$\begin{aligned}\frac{\mathrm{d}t_{\text{s}}}{\mathrm{d}\alpha_{\text{A}}} = \frac{d\sin\alpha_{\text{A}}}{u\cos^2\alpha_{\text{A}}} - \frac{d}{v\cos^2\alpha_{\text{A}}} &= 0\,,\\ \frac{d(v\sin\alpha_{\text{A}} - u)}{uv\cos^2\alpha_{\text{A}}} &= 0\,,\\ \sin\alpha_{\text{A}} &= \frac{u}{v}\,.\end{aligned}$$

[1]Since the velocity of the tractor is constant in all directions, motion in a straight line is the most time-efficient.

$$
\begin{aligned}
\frac{\mathrm{d}t_\mathrm{s}}{\mathrm{d}\alpha_\mathrm{B}} &= \frac{d\sin\alpha_\mathrm{B}}{u\cos^2\alpha_\mathrm{B}} - \frac{d}{v\cos^2\alpha_\mathrm{B}} = 0\,,\\
\frac{d(v\sin\alpha_\mathrm{B} - u)}{uv\cos^2\alpha_\mathrm{B}} &= 0\,,\\
\sin\alpha_\mathrm{B} &= \frac{u}{v}\,.
\end{aligned}
$$

This result may be interpreted with knowledge from optics: the tractor must approach the boundary field-driveway at a critical angle. Since α_A and α_B are equal, from now on, we'll denote $\alpha_\mathrm{A} = \alpha_\mathrm{B} = \alpha$. From relations between goniometric functions $\cos\alpha = \sqrt{1-(u/v)^2}$ and $\tan\alpha = (u/v)/\sqrt{1-(u/v)^2}$, which we substitute into the equation (3),

$$
\begin{aligned}
t_\mathrm{s} &= \frac{2d}{u\cos\alpha} + \frac{r - 2d\tan\alpha}{v} = \frac{r}{v} + 2d\frac{v - u\sin\alpha}{uv\cos\alpha}\,,\\
t_\mathrm{s} &= \frac{r}{v} + 2d\frac{v - \frac{u^2}{v}}{u\sqrt{v^2-u^2}}\,,\\
t_\mathrm{s} &= \frac{r}{v} + \frac{2d}{uv}\sqrt{v^2-u^2}\,.
\end{aligned}
$$

From the condition in the problem statement, we know that $t_\mathrm{p} = t_\mathrm{s} = t$, so

$$
\begin{aligned}
\frac{r}{u} &= \frac{r}{v} + \frac{2d}{uv}\sqrt{v^2-u^2}\,,\\
r\,(v-u) &= 2d\sqrt{v^2-u^2}\,,\\
\frac{d}{r} &= \frac{1}{2}\sqrt{\frac{v-u}{v+u}}\,.
\end{aligned}
$$

If Jindra's ride from A to B takes the same time directly through the field as with the detour on the driveway, then the ratio of distances d and r satisfies $d/r = 1/2\sqrt{(v-u)/(v+u)}$.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem FB ... dome

The hemispherical dome of an observatory has a diameter $d = 20\,\mathrm{m}$ and mass $m = 200\,\mathrm{t}$, distributed uniformly. Find the minimal power of motors moving the dome which is needed to turn the dome by 180° in $t = 30\,\mathrm{s}$. The dome is sitting frictionlessly on track bearings. At the start and at the end, the dome must be stationary. *Dodo likes to observe the sky.*

Let us take a look at a different problem first. If we are given the maximal power, the fastest way to turn the dome is to accelerate it with this maximal power half of the time and decelerate it in the second half of the time (also with maximal power). That means it rotates with the maximum possible angular velocity at each point in time, so it turns in the shortest possible time. The desired power is the one which turns the dome in the given time t (we are turning it in this fastest way). Therefore, we need to find the power required to turn the dome by the angle $\Phi = \pi/2 = 90^\circ$ in $t/2 = \tau = 15\,\mathrm{s}$ with constant acceleration.

Since we have a constant power P, the kinetic energy satisfies

$$E_\mathrm{k} = \frac{1}{2} I \omega^2 = Pt\,,$$

where I is the moment of inertia of a hemisphere and ω is its instantaneous angular velocity at time t. We express the angular velocity and after integration by time from the beginning to the half-turn, we have

$$\omega = \sqrt{\frac{2Pt}{I}}\,,$$

$$\Phi = \int_0^t \omega \,\mathrm{d}t = \int_0^\tau \sqrt{\frac{2Pt}{I}} \mathrm{d}t = \frac{2}{3}\sqrt{\frac{2P}{I}} \tau^{\frac{3}{2}}\,.$$

From there, we can express the desired power as

$$P = \frac{9\Phi^2 I}{8\tau^3} = \frac{9\Phi^2 I}{t^3}\,.$$

Now we need only to find the moment of inertia of the hemisphere. It can be calculated as half of the moment of inertia of a sphere (not a ball, by sphere we mean only the surface, or rather a thin layer underneath it), since it's cut in half symmetrically (i.e. the upper and lower hemispheres have the same moments of inertia with respect to the given axis). The hemisphere also has half of the weight of a whole sphere. Therefore

$$I = \frac{2}{3} mR^2 = \frac{1}{6} md^2\,,$$

which, after substitution into the expression for power, gives us

$$P = \frac{3\Phi^2 md^2}{2t^3} = \frac{3\pi^2 md^2}{8t^3} \doteq 11.0\,\mathrm{kW}\,.$$

The power required to turn the hemisphere in the given time is $P = 11.0\,\mathrm{kW}$.

Jozef Lipták
`liptak.j@fykos.cz`

Problem FC ... light please

On a ceiling at a height $h = 2.5\,\mathrm{m}$, there is a light bulb that shines isotropically into the whole space below it with a luminous flux $\Phi = 1400\,\mathrm{lm}$. How large is the area on the ground where the illuminance is greater than $E_0 = 25\,\mathrm{lx}$? *Danka needs better lights at the dorm.*

Illuminance at a distance r from the source is given by the formula

$$E = \frac{I}{r^2} \cos\alpha\,,$$

where I is the luminous intensity of the source and α is the angle between the incident beam and the normal to the illuminated area. For the luminous intensity, we have

$$I = \frac{\Phi}{\Theta}\,,$$

where Θ is the solid angle at which light propagates from the source. In our case, $\Theta = 2\pi$. The illuminance is then $E = \frac{\Phi \cos\alpha}{2\pi r^2}$. The distance r is simply expressed from a right triangle as $r = \frac{h}{\cos\alpha}$. Then, the condition for illumination can be written as

$$\frac{\Phi \cos^3\alpha}{2\pi h^2} > E_0 \,.$$

From there, we get the condition $\cos\alpha > \sqrt[3]{\frac{E_0 h^2 2\pi}{\Phi}}$, so the maximum angle is $\alpha_m \approx 27.3\,^{\circ}$. Then, the area of the circle is $S = \pi x^2$, where $x = h \tan\alpha_m$, so $S = \pi h^2 \tan^2\alpha \doteq 5.2\,\mathrm{m}^2$. The area on the ground where the illuminance is greater than $25\,\mathrm{lx}$ is about $5.2\,\mathrm{m}^2$.

Daniela Pittnerová
daniela@fykos.cz

Problem FD ... electron-y

What are the possible values of the total electron spin quantum number of a neutral atom of nitrogen? For each option, it is possible to find the most energetically favourable configuration. State the values of the spin in the order that corresponds to increasing energy of these confugurations. *Dodo is breaking Hund's rules.*

The spin quantum number of an electron is 1/2 and an atom of nitrogen has 7 electrons. The total value of the electron spin of the whole atom can reach only the values which we can get by choosing the signs in the expression

$$\left|\frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2}\right| ,$$

so after combining the spins, we get values 1/2, 3/2, 5/2 and 7/2. Pauli's principle implies that a full orbital has spin 0. According to Hund's rules, the lowest energy state of the nitrogen atom is $1s^2 2s^2 2p_x^1 2p_y^1 2p_z^1$, where in p orbitals, all of the spins are oriented the same direction (WLOG[2] up). Then, the nitrogen has a total spin 3/2, which we find as the difference between the number of electrons with upward spin and the number of electrons with downward spin. For the other options, we need to increase the energy. For the total spin 1/2, a state with only one unpaired electron works - for instance, $1s^2 2s^2 2p_x^2 2p_y^1 2p_z^0$; another option is to take the ground state and just flip the spin of one of the unpaired electrons. For the other options, we can't use only the orbitals that are full in the ground state, because we need to have at least 5 unpaired electrons. For a spin 5/2, we have to open the orbital $3s$ and use the state $1s^2 2s^1 2p_x^1 2p_y^1 2p_z^1 3s^1$, which will have much higher energy. The energetically worst is the lowest-energy state with the

[2] Without Loss Of Generality

spin 7/2, which requires us to open one more orbital $3p$; all seven lowest-energy orbitals need to have unpaired electrons, so the configuration is $1s^1 2s^1 2p_x^1 2p_y^1 2p_z^1 3s^1 3p^1$.

$$S = \frac{3}{2}: \quad \underset{1s}{\boxed{\uparrow\downarrow}} \quad \underset{2s}{\boxed{\uparrow\downarrow}} \quad \underset{2p}{\boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}}$$

$$S = \frac{1}{2}: \quad \underset{1s}{\boxed{\uparrow\downarrow}} \quad \underset{2s}{\boxed{\uparrow\downarrow}} \quad \underset{2p}{\boxed{\uparrow\downarrow}\boxed{\uparrow}\boxed{}}$$

$$S = \frac{5}{2}: \quad \underset{1s}{\boxed{\uparrow\downarrow}} \quad \underset{2s}{\boxed{\uparrow}} \quad \underset{2p}{\boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}} \quad \underset{3s}{\boxed{\uparrow}}$$

$$S = \frac{7}{2}: \quad \underset{1s}{\boxed{\uparrow}} \quad \underset{2s}{\boxed{\uparrow}} \quad \underset{2p}{\boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}} \quad \underset{3s}{\boxed{\uparrow}} \quad \underset{3p}{\boxed{\uparrow}\boxed{}\boxed{}}$$

Jozef Lipták
`liptak.j@fykos.cz`

Problem FE ... thirteen wells

Jáchym usurped all twelve wells from the previous problems and added one more, the thirteenth well. Then, to each well, he poured $V_0 = 169.00\,\mathrm{m}^3$ of one of thirteen different liquids. The last, thirteenth well, was full of blood. The ritual of invoking demons could begin. First, Jáchym transferred a volume $V = 13.00\,\mathrm{m}^3$ from the thirteenth well to the first one. After proper mixing, he transferred a volume V from the first well back to the thirteenth well. Then, he mixed up the liquid in the thirteenth well again and repeated the process with the second well, the third well and so on, up to the 12-th well. Altogether, there were 24 pourings. In the end, Jáchym measured the volume fractions of all liquids in the thirteenth well and multiplied these values. What was the result?

Jáchym was inspired by the problem "thirteen barrels" from the 7th Online Physics Brawl.

Let's substitute $k = \frac{V}{V_0}$ and assume from now on that the volume in each well is 1. The composition of the liquid in the thirteenth well after i steps is described by a vector $\mathbf{x}_i$. The concentration of the j-th liquid is expressed by the number x_i^j, where the blood has the index 0. At the begining (before all pourings), we may write

$$x_0^0 = 1\,,$$
$$x_0^j = 0 \quad \forall j > 0\,.$$

The convention used in this problem is as follows: We have thirteen vectors $\mathbf{x}$, which are indexed from $\mathbf{x}_0$ for the initial state to $\mathbf{x}_{12}$ for the final state. Each of these vectors has thirteen independent components denoted by upper indices (the notation is usually used for powers, but not in this case), so x_i^j is the j-th component of the vector $\mathbf{x}_i$ and it's just a number (a scalar).

Now, we will proceed by induction. After $i-1$ steps, we are in the state described by the vector $\mathbf{x}_{i-1}$. We take the volume $V = kV_0$ from the thirteenth well, represented by the expression $k\mathbf{x}_{i-1}$, and pour it into the i-th well. This well was full of the i-th liquid, so we can describe its initial composition by the vector $\mathbf{e}_i$, whose i-th component is 1 and all other components are 0. After the mixing of the initial liquid and new liquid from the 13-th well, we get the mixture described by $\mathbf{e}_i + k\mathbf{x}_{i-1}$.

The volume of the i-th well is now $V' = V + V_0 = (1+k)\,V_0$. From that, we take the volume

$$V = kV_0 = \frac{k}{1+k}V'\,.$$

Into the thirteenth well, we pour the liquid described by

$$\frac{k}{1+k}\left(\mathbf{e}_i + k\mathbf{x}_{i-1}\right)\,.$$

Before this step, there was the mixture $\mathbf{x}_{i-1}$ in the 13-th well; after taking the volume kV_0, there remained $(1-k)\,\mathbf{x}_{i-1}$, and we pour the mixture described by the expression above into it. The resulting mixture is

$$\mathbf{x}_i = (1-k)\,\mathbf{x}_{i-1} + \frac{k}{1+k}\left(\mathbf{e}_i + k\mathbf{x}_{i-1}\right) = \frac{\mathbf{x}_{i-1} + k\mathbf{e}_i}{1+k}\,.$$

We just expressed the change in the composition of the mixture in the 13-th well for one step. It was more or less a trivial application of the mixing equation, but we worked with 13 components at once. The notation we used may seem a bit complicated, but that is because we worked quite generally. On the other hand, we now have a formula that applies from the beginning to the end of the ritual.

Notice that the i-th step is the only event in which the i-th liquid is added to the thirteenth well. The concentration of the i-th liquid is now just decreasing $(1+k)$ times in each step. Now, we can write the final composition of the mixture in the thirteenth well

$$\mathbf{x}_{12} = \frac{\mathbf{e}_0}{(1+k)^{12}} + \frac{k}{1+k}\sum_{j=1}^{12}\frac{1}{(1+k)^{12-j}}\mathbf{e}_j\,.$$

We want the product of volume fractions, i.e. the product of the expressions before the unit vectors $\mathbf{e}$ in the expression above. Let's denote the result of this problem by P. Then, we may write

$$P = \frac{1}{(1+k)^{12}}\cdot\left(\frac{k}{1+k}\right)^{12}\cdot\prod_{j=1}^{12}\frac{1}{(1+k)^{12-j}} = \frac{k^{12}}{(1+k)^{24}}\cdot\prod_{j=0}^{11}\frac{1}{(1+k)^{j}} =$$
$$= \frac{k^{12}}{(1+k)^{90}} = V^{12}V_0^{78}\left(V+V_0\right)^{-90} \doteq 5.45\cdot 10^{-17}\,.$$

Jáchym Bártík
tuaki@fykos.cz

Problem FF ... boiling

Mikuláš has a well, but he is too lazy to pull water out of it by hand. Therefore, he bought a pump with an electric motor, whose efficiency η does not depend on the voltage or the current. What is the minimal depth of the well such that it would be more efficient to convert water into steam instead of pumping normally? The original temperature of the water is T and it would be boiled by a spiral with resistance R. The resistance per unit length of the power line that connects the spiral with the voltage source on the ground is λ. We expect a general result expressed using the quantities specified in the problem statement and generally known constants. *Jáchym remembered the problem DA from the 11. FYKOSí Fyziklání.*

Let's denote the depth of the well by h. Then the resistance of the power line is $h\lambda$. The power line and the spiral create a series electrical circuit with resistors and a source. As we want to minimise h, we should choose the voltage U on the source in a way that makes the heating the most efficient. The current in the network is

$$I = \frac{U}{R + h\lambda}\,,$$

and the thermal power of the spiral is

$$P_{\mathrm{d}} = U_R I = RI^2 = \frac{R}{(R + h\lambda)^2} U^2\,.$$

The corresponding electrical power of the engine is

$$P_{\mathrm{m}} = \eta U I = \eta U I = \eta \frac{1}{R + h\lambda} U^2 = \eta \frac{R + h\lambda}{R} P_{\mathrm{d}}\,.$$

Now, we have to find the mass of water that we move up per unit time with this power. In the case of the pump, it's quite easy, because the potential energy is mgh, so the mass flow rate is

$$q_{\mathrm{m}} = \frac{P_{\mathrm{m}}}{gh}\,.$$

To boil the water, we have to supply the energy $m\left(c\left(T_{\mathrm{v}} - T\right) + l_{\mathrm{v}}\right)$, where T_{v} is the boiling point, c is the specific heat capacity and l_{v} is the specific latent heat of vaporisation. Altogether, we get

$$q_{\mathrm{d}} = \frac{P_{\mathrm{d}}}{c\left(T_{\mathrm{v}} - T\right) + l_{\mathrm{v}}}\,.$$

We want to know when the mass flow rate is greater for boiling. This corresponds to the condition $q_{\mathrm{d}} \geq q_{\mathrm{m}}$, or

$$\frac{P_{\mathrm{d}}}{c\left(T_{\mathrm{v}} - T\right) + l_{\mathrm{v}}} \geq \frac{P_{\mathrm{m}}}{gh}\,,$$

$$\frac{Rg}{\eta\left(c\left(T_{\mathrm{v}} - T\right) + l_{\mathrm{v}}\right)} \frac{h}{R + h\lambda} \geq 1\,.$$

Now, we can see that the expression on the left is an increasing function that passes through the origin. That implies that equality holds for no more than one value h_0, and for all possible $h > h_0$, the condition above holds. Finally, we express the result

$$h_0 = \frac{R\eta\left(c\left(T_{\mathrm{v}} - T\right) + l_{\mathrm{v}}\right)}{Rg - \lambda\eta\left(c\left(T_{\mathrm{v}} - T\right) + l_{\mathrm{v}}\right)} = \left(\frac{g}{\eta\left(c\left(T_{\mathrm{v}} - T\right) + l_{\mathrm{v}}\right)} - \frac{\lambda}{R}\right)^{-1}\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem FG ... inverse well

Jáchym has an inverse well – water is constantly being replenished into it with a constant mass flow rate $q = 17.0\,\mathrm{kg\cdot min^{-1}}$ by a stream with a circular cross section. At what depth does continuous flow become unstable and break down into droplets with a radius $r = 2.5\,\mathrm{mm}$? The velocity of the water at the top of the well and air resistance are negligible.

Jáchym likes alternative wells.

Nature "tries" to minimise the total energy of any system. The droplets are being formed at a moment when their total energy would be lower than that of the stream. The surface energy is directly proportional to the surface area, so it is sufficient to minimise the area. A droplet with a radius r has surface area $S_\mathrm{k} = 4\pi r^2$. The droplet corresponds to a part of the stream with the same volume, which can be approximated as a cylinder with a height h and cross-sectional area S. Then, since it has the same volume as the droplet,

$$hS = \frac{4}{3}\pi r^3 .$$

Both bases of the cylinder are adjacent to other parts of the stream, so the bases don't contribute to the total surface energy. The surface area of the corresponding open cylinder is then

$$S_\mathrm{v} = 2\sqrt{\pi S}h = \frac{8\pi}{3}\sqrt{\frac{\pi}{S}}r^3 .$$

For $S_\mathrm{k} < S_\mathrm{v}$, the droplets are energetically more advantageous than the stream. For this case, we get

$$4\pi r^2 = \frac{8\pi}{3}\sqrt{\frac{\pi}{S}}r^3 ,$$

$$S = \frac{4\pi}{9}r^2 .$$

Now we know the cross-sectional area of the stream at the moment of the split and we need the depth at which the split happens. The acceleration of the water is equal to the gravity of Earth, so in a time t, it falls by

$$x = \frac{1}{2}gt^2 .$$

The velocity at the time t is $v = gt$. The cross-sectional area S is related to the velocity and the volumetric flow rate as $vS = q_V = q/\varrho$, where ϱ is the density of water. By combining these equations, we get

$$x = \frac{v^2}{2g} = \frac{q^2}{2g\varrho^2 S^2} = \frac{81q^2}{32\pi^2 g\varrho^2 r^4} \doteq 54\,\mathrm{m} .$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem FH ... of course it's drinking water

Jáchym has a well and he constantly draws water from it with a volumetric flow rate $q_V = 0.2\,\mathrm{l{\cdot}s^{-1}}$. Water is being constantly replenished into the well so the total volume stays constant at $V = 68\,\mathrm{m^3}$. However, Jáchym accidentally dropped a piece of radioactive bread into the well, which dissipated perfectly in the water. The bread contained $3.0 \cdot 10^{15}$ radioactive isotopes with a half-life $T = 69\,\mathrm{h}$. Jáchym decided to ignore it and continued to draw water at the original rate. How long does it take until the radioactive activity of the well drops below $A = 1\,900\,\mathrm{s^{-1}}$?

Originally, this should have been about Danka's hair, but Jáchym said it will be about a well.

Radioactive decay obeys the equation

$$N_\mathrm{r} = N_0 \mathrm{e}^{-\lambda_\mathrm{r} t}\,,$$

where N_0 is the initial number of particles and $\lambda_\mathrm{r} = \frac{\ln 2}{T}$ is the decay constant. The number of particles that decay in a time $\mathrm{d}t$ is then

$$-\mathrm{d}N_\mathrm{r} = -\dot{N}_\mathrm{r}\mathrm{d}t = \lambda_\mathrm{r} N_0 \mathrm{e}^{-\lambda_\mathrm{r} t}\mathrm{d}t = \lambda_\mathrm{r} N \mathrm{d}t\,.$$

However, in our case, the radioactive solution is also being diluted by the clean water coming to the well. After the time $\mathrm{d}t$, the volume $q_V\mathrm{d}t$ will flow through the well. The number of radioactive particles leaving the well that way is

$$\frac{q_V \mathrm{d}t}{V} N = -\mathrm{d}N_\mathrm{v}\,.$$

Therefore, the equation describing the change in the number of radioactive particles in the well is

$$\mathrm{d}N = \mathrm{d}N_\mathrm{r} + \mathrm{d}N_\mathrm{v} = -\left(\lambda_\mathrm{r} + \frac{q_V}{V}\right) N \mathrm{d}t = -\lambda N \mathrm{d}t\,.$$

We now see that the number of particles in the well will again be an exponential function, but with a different constant. The activity can be calculated as $A = \lambda_\mathrm{r} N$, or

$$A = \lambda_\mathrm{r} N_0 \mathrm{e}^{-\lambda t}\,.$$

From this, we get the resulting time

$$t = -\frac{1}{\lambda} \ln\left(\frac{A}{\lambda_\mathrm{r} N_0}\right) = \frac{1}{\frac{\ln 2}{T} + \frac{q_V}{V}} \ln\left(\frac{N_0}{A}\frac{\ln 2}{T}\right) \doteq 740\,\mathrm{hod}\,.$$

Jáchym Bártík
`tuaki@fykos.cz`

Problem GA ... hole in a bucket

Dodo has a well with a bucket, which has a cylindrical shape - its height is $h_0 = 32\,\mathrm{cm}$, the radius of its base is $r = 12\,\mathrm{cm}$ and its weight is $m = 2.7\,\mathrm{kg}$. At the bottom of the bucket, there is a hole with a cross-section $S = 1.0\,\mathrm{cm^2}$. Dodo pulls the bucket up from a depth $H = 25\,\mathrm{m}$ at a constant speed $v = 0.40\,\mathrm{m{\cdot}s^{-1}}$. Compared to the situation if the bucket wasn't leaky, how many times less efficient is this procedure? We are asking about the ratio of works which Dodo *has to perform to pull a unit amount of water out of the well in both cases.*

Jáchym was eating soup with a fork.

Water from the bucket flows out with velocity $\sqrt{2hg}$, where h is the current water level. The volumetric flow rate is $q = S\sqrt{2hg}$, which can also be written as $-\dot{V}$, where $V = \pi r^2 h$ is the volume of water in the bucket. Hence, we get the equation

$$\pi r^2 \dot{h} = -S\sqrt{2hg}\,,$$

which has the solution

$$h = \left(\sqrt{h_0} - \frac{S}{\pi r^2}\sqrt{\frac{g}{2}}t\right)^2 .$$

The time Dodo needs to pull the bucket out is simply calculated as

$$\tau = \frac{H}{v}\,.$$

By substituting $t = \tau$ into the previous equation, we verify that all the water does not run out on the way up, and we find out that the water level in the leaky bucket when Dodo pulls it out is $h_1 = 6.75\,\mathrm{cm}$.

The total weight of the bucket with water is $m + \pi r^2 h$. From this, we get the force with which Dodo must pull

$$F = \left(m + \pi r^2 h\varrho\right) g\,.$$

Then, the power is $P = Fv$. Work is the integral of power over time

$$W_1 = \int_0^\tau P\,\mathrm{d}t = mgv\tau + \pi r^2 \varrho g v \int_0^\tau h(t)\,\mathrm{d}t = mgv\tau + \pi r^2\varrho g v\left(-\frac{\pi r^2}{S}\sqrt{\frac{2}{g}}\right)\frac{1}{3}\left[h^{\frac{3}{2}}(t)\right]_0^\tau =$$
$$= mgH + \frac{\pi^2 r^4 \varrho v\sqrt{2g}}{3S}\left(h_0^{\frac{3}{2}} - h_1^{\frac{3}{2}}\right) = 2.638\,\mathrm{kJ}\,.$$

The resulting volume of water that Dodo pulls up is $V_1 = \pi r^2 h_1$.

In the second case, the situation is much simpler — the force is the same all the time, so we calculate the work as

$$W_2 = \left(m + \pi r^2 h_0 \varrho\right) gH = 4.211\,\mathrm{kJ}\,.$$

The volume of water also remains the same, specifically $V_2 = \pi r^2 h_0$.

The solution is the ratio

$$\frac{W_1}{V_1}\frac{V_2}{W_2} = \frac{W_1}{W_2}\frac{h_0}{h_1} \doteq 2.97\,.$$

If Dodo bought a new bucket, he would be nearly three times more efficient in pumping water from the well.

Jáchym Bártík
`tuaki@fykos.cz`

Problem GB ... excess heat

Geothermal energy comes from the decay of radioactive elements, from tidal deformations of the Earth and from the residual heat released by differentiation of the layers of Earth. The Earth has mass $M = 5.97 \cdot 10^{24}\,\mathrm{kg}$ and radius $R = 6.38 \cdot 10^{3}\,\mathrm{km}$. Assume that at the beginning of its existence, it was a homogeneous sphere. After differentiation, a metallic (predominantly iron) core with a radius $r_\mathrm{j} = 3.50 \cdot 10^{3}\,\mathrm{km}$ and a density $\varrho_\mathrm{j} = 13\,000\,\mathrm{kg{\cdot}m^{-3}}$ formed within the Earth. The remainder of the Earth is the mantle, with a constant density. Find the heat released by this differentiation. *Jindra felt like his soles were on fire.*

In the problem statement, it was mentioned that we should assume a homogeneous density of the mantle. First of all, though, we have to calculate it. The volume of a spherical shell with an inner radius r_1 and outer radius r_2 is

$$V_\mathrm{sl} = \frac{4}{3}\pi\left(r_2^3 - r_1^3\right).$$

The mass of the whole Earth is $M = 5.97 \cdot 10^{24}\,\mathrm{kg}$. The mass of the Earth's core can be calculated from its radius and density as

$$M_\mathrm{j} = \frac{4}{3}\pi r_j^3 \varrho_\mathrm{j} = 2.335 \cdot 10^{24}\,\mathrm{kg}.$$

Then, the average density of the Earth's mantle $\varrho_{\ p}$ can be expressed as

$$\varrho_\mathrm{p} = \frac{M - M_\mathrm{j}}{\frac{4}{3}\pi\left(R^3 - r_\mathrm{j}^3\right)},$$
$$\varrho_\mathrm{p} = 4.003 \cdot 10^{3}\,\mathrm{kg{\cdot}m^{-3}}.$$

Now let's move on to the calculation of energy. At the beginning, the Earth is a homogeneous sphere, so its gravitational potential energy is

$$E_1 = -\frac{3GM^2}{5R},$$
$$E_1 = -2.237 \cdot 10^{32}\,\mathrm{J}.$$

After differentiation, the Earth divides into a core region and a mantle region. Its potential energy is equal to the sum of the potential energy of the core and the potential energy of the mantle. According to the shell theorem, the gravitational forces acting on a mass point within a homogeneous spherical shell cancel each other out. Thus, a mass located at a distance r from the centre of the Earth is affected only by gravitational force from mass below this radius r. In other words, the potential energy of the Earth's core $E_{\ j}$ is not affected by the presence of the mantle and is calculated as

$$E_\mathrm{j} = -\frac{3GM_\mathrm{j}^2}{5r_\mathrm{j}} = -6.237 \cdot 10^{31}\,\mathrm{J}.$$

The potential energy of the Earth's mantle can be calculated using an integral. Let's place layers of the mantle on the Earth's core until we build the entire Earth. Suppose that we have already created an "Earth seed" with a radius $r > r_{\ j}$. This means that in the middle, there is the core, and above it is part of the shell with width $r - r_\mathrm{j}$. If we bring an infinitesimally thin spherical shell with a mass $\mathrm{d}m$ from infinity, the potential energy changes by

$$\mathrm{d}E = -\frac{G\frac{4}{3}\pi r_\mathrm{j}^3\varrho_\mathrm{j}\mathrm{d}m}{r} - \frac{G\frac{4}{3}\pi\left(r^3 - r_\mathrm{j}^3\right)\varrho_\mathrm{p}\mathrm{d}m}{r}.$$

We can substitute $\mathrm{d}m = 4\pi\varrho_\mathrm{p}r^2\mathrm{d}r$ and we get

$$\mathrm{d}E = -\frac{16}{3}G\pi^2 r_\mathrm{j}^3\varrho_\mathrm{j}\varrho_\mathrm{p}r\mathrm{d}r - \frac{16}{3}G\pi^2\left(r^3 - r_\mathrm{j}^3\right)\varrho_\mathrm{p}^2 r\mathrm{d}r,$$
$$\mathrm{d}E = \frac{16}{3}G\pi^2 r_\mathrm{j}^3\varrho_\mathrm{p}\left(\varrho_\mathrm{p} - \varrho_\mathrm{j}\right)r\mathrm{d}r - \frac{16}{3}G\pi^2\varrho_\mathrm{p}^2 r^4\mathrm{d}r$$

Since the Earth's mantle extends between the radii r_j and R, its gravitational potential energy is calculated as

$$E_\mathrm{p} = \frac{16}{3}G\pi^2 r_\mathrm{j}^3\varrho_\mathrm{p}\left(\varrho_\mathrm{p} - \varrho_\mathrm{j}\right)\int_{r_\mathrm{j}}^{R} r\mathrm{d}r - \frac{16}{3}G\pi^2\varrho_\mathrm{p}^2\int_{r_\mathrm{j}}^{R} r^4\mathrm{d}r,$$
$$E_\mathrm{p} = \frac{8}{3}G\pi^2 r_\mathrm{j}^3\varrho_\mathrm{p}\left(\varrho_\mathrm{p} - \varrho_\mathrm{j}\right)\left(R^2 - r_\mathrm{j}^2\right) - \frac{16}{15}G\pi^2\varrho_\mathrm{p}^2\left(R^5 - r_\mathrm{j}^5\right),$$
$$E_\mathrm{p} = -1.903\cdot 10^{32}\,\mathrm{J}.$$

The total potential energy E_2 of the Earth after differentiation is the sum of the potential energy of the core $E_{\ j}$ and the potential energy of the mantle $E_{\ p}$

$$E_2 = E_\mathrm{j} + E_\mathrm{p} = -2.526\cdot 10^{32}\,\mathrm{J}.$$

At the beginning, the Earth had more potential energy than after differentiation. The heat Q released by differentiation is their difference

$$Q = E_1 - E_2 = 2.9\cdot 10^{31}\,\mathrm{J}.$$

The released heat is $2.9\cdot 10^{31}$ J, which corresponds to "calorific value" 5 MJ·kg−1.

Jindřich Jelínek
`jjelinek@fykos.cz`

Problem GC ... parabolic rays

Your task is to find out how the refractive index of the atmosphere should depend on the height y above the surface, if we want the rays in the atmosphere to travel along parabolic trajectories $y = x^2$. Neglect the curvature of Earth. *Jurčo wants to see sideways.*

Using Snell's law, we should first realise that the refractive index must increase with height. The material (atmosphere) can be divided into thin horizontal layers with a constant refractive index. The beam will tilt by a small angle $\mathrm{d}\vartheta$ when moving up a layer. For two adjacent layers of the atmosphere, we can write Snell's law as

$$(n + \mathrm{d}n)\sin(\vartheta + \mathrm{d}\vartheta) = n\sin\vartheta\,,$$
$$n\sin\vartheta + n\cos\vartheta\mathrm{d}\vartheta + \mathrm{d}n\sin\vartheta = n\sin\vartheta\,,$$

from which we have

$$\frac{\mathrm{d}n}{n} = -\frac{\mathrm{d}\vartheta}{\tan\vartheta}\,.$$

The tangent of the angle between the beam and the vertical is obtained from the tangent of the parabola $y = x^2$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x = \tan(\pi/2 - \vartheta) = \cot\vartheta = \frac{1}{\tan\vartheta}\,.$$

By further differentiating this equation, we get the relationship between $\mathrm{d}\vartheta$ and $\mathrm{d}x$

$$2\mathrm{d}x = -\frac{\mathrm{d}\vartheta}{\sin^2\vartheta}\,,$$

where the sine squared can be expressed as

$$\sin^2\vartheta = \frac{1}{\cot^2\vartheta + 1}$$
$$\sin^2\vartheta = \frac{1}{4x^2 + 1} = \frac{1}{4y + 1}\,.$$

After substituting $\sin^2\vartheta$ and $\mathrm{d}\vartheta$ into our form of Snell's law, we have

$$\frac{\mathrm{d}n}{n} = \frac{1}{2}\frac{4\mathrm{d}y}{4y + 1}\,,$$

and from this, we get the resulting dependence of n on y

$$n = n_0\sqrt{\frac{4y + 1}{4y_0 + 1}}\,.$$

Róbert Jurčo
robert.jurco@fykos.cz

FYKOS
UK, Matematicko-fyzikální fakulta
Ústav teoretické fyziky
V Holešovičkách 2
180 00 Praha 8

www: `http://fykos.cz`
e-mail: `fykos@fykos.cz`

FYKOS is also on Facebook
http://www.facebook.com/FYKOS

FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Public Relations Office activities and is supported by Institute of Theoretical Physics, MFF UK, his employees and The Union of Czech Mathematicians and Physicists.